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Contests with Ambiguity[1]

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Abstract

The paper examines the effect of ambiguity on contests where multiple parties expend resources to win a prize. We develop a model where contenders perceive ambiguity about their opponents' strategies and determine how perceptions of ambiguity and attitudes to ambiguity affect equilibrium choice. The paper also investigates how equilibrium under ambiguity is related to behavior where contenders have expected utility preferences. Our model can explain experimental results such as overbidding and overspreading relative to Nash predictions.

1 Introduction

Many important economic interactions can be modelled as contests. Applications of contest theory include rent-seeking, beauty contests and influence activities, internal labor market tournaments, litigation, R&D and patent races, financing of public goods, marketing campaigns, military conflict, political competition, and sporting contests. Theoretical and empirical advances in the area have led to a better understanding of the strategic forces and trade-offs in these important economic environments and to recommendations for improving upon outcomes. However, there is a stark dissonance between a number of standard theoretical results and their empirical tests which jeopardizes the practical import of the theory.

A flurry of experimental studies[1] reveal, in their vast majority, that average expenditure to win the prize is significantly higher than the Nash prediction (commonly referred to as overbidding) and the variance of expenditure across experimental subjects is considerable (overspreading). In some experiments, the extent of overbidding is so prominent that the average earnings are negative. A number of possible rationalizations have been put forth. Explanations of overbidding include hypotheses that experimental subjects

- derive a non-monetary utility from winning, on top of monetary incentives to win a prize (e.g., Sheremeta, 2010, Chen et al., 2011),
- exhibit behavior sensitive to the experimental design (Chowdhury et al., 2012),
- have spiteful preferences and inequality aversion (e.g., Herrmann and Orzen, 2008; Bartling et al., 2009; Balafoutas et al., 2012),
- have a predisposition to make mistakes (Potters et al., 1998, Lim et al., 2012),
- rely on non-linear probability weighting to make their bids (Baharad and Nitzan, 2008, Amaldoss and Rapoport, 2009, Duffy and Matros, 2012), and
- exhibit loss aversion (Kong, 2008).

[1] See Dechenaux et al. (2012) for an extensive survey of the experimenal research on contests.

Differences in these behavioral traits can also explain, at least in theory, a part of the large variation in expenditure of experimental subjects. The overspreading has also been linked to variation in risk aversion and demographic characteristics of experimental subjects.

We propose an alternative explanation for these phenomena by developing a model where participants in a contest may perceive ambiguity about their opponents' strategies and determining how perceptions of ambiguity and attitudes to ambiguity affect equilibrium behavior. A decision-maker has an ambiguous belief if it is not precise enough to be represented by a single (objective or subjective) probability distribution as in the expected utility model (Savage, 1954) or, more generally, as in the probabilistically sophisticated model (Machina and Schmeidler, 1992). In his pioneering study, Ellsberg (1961) argued that individuals will exhibit behavior that reveals preferences which differentiate amongst risk (known probabilities) and ambiguity (unknown probabilities). The prevalence of Ellsberg-type behavior in experimental and naturally occurring settings[2] and the inability of Savages' subjective expected utility theory to explain it have stimulated efforts to develop and axiomatize alternative models of decision-making (Schmeidler, 1989, Gilboa and Schmeidler, 1989, Bewley, 2002, Ghirardato and Marinacci, 2002, Epstein and Schneider, 2003, Klibanoff et al., 2005, Maccheroni et al., 2006). Applications of these models have led to novel insights and prescriptions for a plethora of economic environments (Gilboa, 2012, Etner et al., 2012).

One of the most popular and promising of these models is Choquet expected utility (henceforth CEU), where decision-makers' beliefs are characterized by non-additive probabilities (or capacities); see Schmeidler (1986, 1989). A CEU decision-maker maximizes the expected value of a utility function with respect to a non-additive belief where the expectation is expressed as a Choquet integral, (Choquet, 1953–4). CEU is a generalization of subjective expected utility. It has the advantage of maintaining the separation of beliefs and outcome evaluation, which makes the theory easily amenable to various applications. To preserve a clear separation between the degree of perceived ambiguity and ambiguity attitude, we further assume CEU preferences with neo-additive capacities (see below for a definition).

[2] For reviews of the literature on ambiguity, see, e.g., Camerer and Weber (1992), Gilboa (2012), and Etner et al. (2012), Machina and Siniscalchi (2013).

Our interest in the effect of ambiguity on behavior in contests is not solely driven by finding a new explanation for overbidding and overspreading. Rather, our primary motivation is that it may be intrinsically difficult for contest participants to attach unique probabilities to the behavior of other contenders. In other words, participants in many real-world contests may perceive ambiguity about their opponents' choices and contenders' attitudes to ambiguity, such as optimism and pessimism, may be at play. To be more specific about pertinence of ambiguity in economic environments, we illustrate with the following examples.

Consider a competition where multiple candidates expend resources to win a political office. Each politician has multiple instruments at her/his disposal to affect the outcome of an election including the amount, timing, and type of advertising[3], campaign promises, and debate strategies. A candidate will condition her/his actions to win the election on expectations of what her/his opponents might do. The standard approach to analyzing this strategic environment hinges upon the assumption that each contender's strategy is conditioned on predictions of the opponents' choices represented by a unique probability distribution and that the equilibrium beliefs are correct. That is, contestants are assumed to behave according to the prescriptions of Nash equilibrium. In reality, a contender may entertain multiple scenarios about the strategies that will be employed by her political opponents. For example, under one scenario a contestant's opponents pursue a relatively negative campaign with a relatively large likelihood while under a different scenario the likelihood of a negative campaign by the opponents is relatively small. In other words, a contender may be unable to assign a unique probability to each course of action by the other contenders.[4]

As another example consider a litigation process where the opposing sides spend resources to affect the outcome of the process in their favor. Does a party to a litigation process have a "clear" idea, in probabilistic sense, about the strategy that will be followed by the opponent? For many cases that are not settled prior to going to court and are not characterized by "regular motions", a considerable amount of ambiguity may be present about strategies that

[3] One of the key decisions in political advertising is the choice of issues which to focus on.

[4] In addition to predicting the opponents' behavior, candidates also need to take into account the behavior of potential voters. This may also involve considerable degree of ambiguity from the perspective of contenders and pundits alike. For example, turnout of different segments of voters may be hard to predict which complicates the task of targeting different fragments of the electorate. Although we don't explicitly model this type of ambiguity, the techniques in the present paper can be rather easily extended to address this scenario.

will be followed by the opposing side.[5]

Finally, consider a typical experiment testing predictions for a lottery game where experimental subjects choose the number of lottery tickets to purchase and a participant's probability of winning the lottery is equal to the ratio of the number of tickets she has purchased to the total number of tickets sold. A subject in this type of experiment is likely to be uncertain about the number of lottery tickets that will be purchased by the other participants. She may entertain a range of possibilities for the number of tickets that are bought by her opponents. Furthermore, it is not at all clear that she will assign a unique probability to each of these possibilities. She may very well contemplate a set of likelihoods for some of the prospects. In other words, the subject's beliefs may be ambiguous. The subjects may not only perceive ambiguity about the opponents' possible play but may also exhibit sensitivity to this ambiguity. An optimistic player (or, equivalently, an ambiguity-loving decision-maker) will expect her opponents to buy a relatively small number of tickets. In contrast, a pessimist (or an ambiguity-averse decision-maker) will expect her opponents to buy a relatively large number of tickets. As a result, an increase in the magnitude of ambiguity may have very different effects on pessimistic and optimistic contenders.

Ambiguity about opponents' possible equilibrium strategies is not restricted to the above examples. Military conflicts, R&D contests, and influence activities may also entail sizeable ambiguity about opponents' behavior.

We develop a model where contenders perceive ambiguity about strategies that are used by their opponents. We prove existence of equilibrium under ambiguity and determine how the degree of ambiguity regarding other participants' strategies and preferences toward ambiguity affect equilibrium behavior. The paper also investigates how equilibrium under ambiguity is related to behavior where contenders have expected utility preferences. Finally, the model in the present paper can also explain overbidding and overspreading relative to Nash prediction that are commonly observed in experimental studies of contests.

[5] The probability of a favorable verdict may also be ambiguous because the litigating parties are likely to have little information about the disposition of the judicial body rendering the verdict.

2 The model

Consider a contest between $n \geq 2$ players. To improve her chances of winning the prize each contestant $i \in \{1, 2, ..., n\}$ chooses action $x_i \in X_i = [\underline{x}_i, \bar{x}_i]$, where $\underline{x}_i \geq 0$ and $\bar{x}_i > 0$. On occasion, we will refer to these actions as effort or expenditure invested in the contest. The assumption that the expenditure to win the contest may be constrained from above accounts for possibilities of budget-constrained participants[6] and for possible exogenous restrictions on the level of expenditures in the contest (e.g., an upper threshold on expenditures set by a "contest designer"). We also model lower bounds on expenditures because in many actual contests there is often a good reason to believe that some contender(s) that decide to participate will spend at least a certain amount on winning the prize. For example, in many presidential elections candidates receive public funds to compete. These serve as a lower bound on the amount that the candidates will spend on their election campaigns. There is an alternative interpretation, which is at least equally plausible, of the lower and upper bounds on expenditures incurred in a context. Those bounds may represent the opponents' beliefs about a player's potential choices. For many competitive environments, it is sensible to expect that players will not anticipate that all of their opponents will choose zero efforts; $\underline{x}_i > 0$ for some or even all i. It is equally reasonable to believe that contenders will have a certain finite upper bound on the opponents' expenditures that will not be exceeded by the other players; $\bar{x}_i < \infty$.

The cost of action x_i is given by x_i and incurred irrespective of whether the contest is won. The probability that contestant i receives the prize, the contest success function (CSF), is given by

$$p_i(x_i; \mathbf{x}_{-i}) = \begin{cases} \frac{h_i(x_i)}{\sum_{j=1}^{n} h_j(x_j)} & \text{if } \exists j \in \{1, ..., n\} \text{ such that } x_j > 0 \\ \frac{1}{n} & \text{if } x_j = 0 \text{ for all } j \in \{1, ..., n\} \end{cases}, \tag{1}$$

where $\mathbf{x}_{-i} \equiv (x_1, ..., x_{(i-1)}, x_{(i+1)}, ..., x_n)$ denotes the vector of action choices by all players except for contestant i. The set of strategy combinations of player i's opponents is denoted

[6] Since resources are scarce, all of the participants in a contest will be budget constained. However, for some or all contenders the budget constraint may be non-binding.

by $\mathbf{X}_{-i}$ and the set of strategy combinations of all players is denoted by $\mathbf{X}$. We also let $\mathbf{x}$ denote the vector of action choices by all participants in the contest; $\mathbf{x} \equiv (x_1, x_2, ..., x_n)$. The function $h_i(\cdot)$ $(i = 1, ..., n)$ is assumed to be increasing in its argument. Under this assumption, p_i is increasing in own action and decreasing in the actions of the opponents. It is also assumed that $h_i(\cdot)$ is concave and twice-continuously differentiable and $h_i(0) = 0$ for all $i = 1, ..., n$. The assumption of concavity of $h_i(\cdot)$ implies that $p_i(x_i; \mathbf{x}_{-i})$ is concave for all $x_i > 0$ and all $\mathbf{x}_{-i}$.

Contestant i's utility function is given by

$$U_i\left(x_i; \sum_{j\neq i} h_j(x_j)\right) = p_i(x_i; \mathbf{x}_{-i}) V_i - x_i, \tag{2}$$

where V_i denotes the value of the prize to contestant i. The assumption that the contenders are risk neutral is made mainly with the purpose of focusing on the effect of ambiguity aversion. Our results carry over to a more general setting with risk averse preferences under appropriate qualifying conditions.

The contest considered in the paper falls into a general category of aggregative games (Selten, 1970; Okuguchi, 1993; Acemoglu and Jensen, 2013; Cornes and Hartley, 2011). The CSF in (1) is decreasing in the aggregate $\left[\sum_{j\neq i} h_j(x_j)\right]$ of the opponents' actions. Hence, the strategic interaction considered in the paper is a game with negative aggregate externalities (Eichberger, Kelsey and Schipper, 2009). The cross-partial derivative of contender i's utility function with respect to own and opponent k's actions is equal to

$$\frac{\partial^2 U_i\left(x_i; \sum_{j\neq i} h_j(x_j)\right)}{\partial x_i \partial x_k} = \frac{\partial^2 p_i(x_i; \mathbf{x}_{-i})}{\partial x_i \partial x_k} V_i,$$

where

$$\frac{\partial^2 p_i(x_i; \mathbf{x}_{-i})}{\partial x_i \partial x_k} = -h_i'(x_i) h_k'(x_k) \frac{\left[\sum_{j\neq i} h_j(x_j)\right] - h_i(x_i)}{\left[\sum_{j=1}^{n} h_j(x_j)\right]^3}.$$

It follows from the expression for $\frac{\partial^2 p_i(x_i; \mathbf{x}_{-i})}{\partial x_i \partial x_k}$ that the marginal benefit of own action is decreasing in opponent k's action when player i's opponents choose relatively large actions

$\left(\sum_{j\neq i} h(x_j) > h(x_i)\right)$ but increasing when the opponents choose relatively small actions $\left(\sum_{j\neq i} h(x_j) < h(x_i)\right)$. In other words, when the aggregate of a player's opponents' efforts is sufficiently large, an increase in any opponent's effort will crowd out the player's effort (the player will partially give in). On the other hand, when the aggregate of a player's opponents' efforts is sufficiently small, the player will respond to an increase in any opponent's effort by increasing her effort (the player will keep up). Thus, this game does not globally exhibit either strategic complementarity or strategic substitutability (Bulow et al., 1985). Note also that when all players choose the same action, the strategies of the players are (local) strategic substitutes.

Suppose that the contenders perceive ambiguity about their opponents' choice of action. This ambiguity is represented by a capacity which reflects the weights a player places on different strategies of the opponents. A capacity is similar to a subjective probability with the exception that it may be non-additive. We restrict our attention to the case where the ambiguity for contestant i is represented by a neo-additive capacity[7] v_i defined on the set of the opponents' strategies $\mathbf{X}_{-i}$:

$$v_i(\varnothing) = 0, v_i(\mathbf{X}_{-i}) = 1, \text{ and } v_i(A) = \delta_i(1-\alpha_i) + (1-\delta_i)\pi_i(A) \text{ for all } \varnothing \subsetneq A \subsetneq \mathbf{X}_{-i}, \quad (3)$$

where $\alpha_i, \delta_i \in [0,1]$ and π_i is a standard probability distribution on $\mathbf{X}_{-i}$. Contestant i has some doubts that the probability distribution $\pi_i(\cdot)$ is the the true probability distribution over the opponents' strategies and this ambiguity is reflected by the parameter δ_i. Parameter α_i characterizes contestant i's ambiguity attitude. The support of a neo-additive capacity v_i is defined by $supp(v_i) = supp(\pi_i)$.[8] We focus on neo-additive capacities because they offer a rather clear-cut separation of ambiguity perception from ambiguity attitude which is not the case for some other theories of ambiguity.[9,10] Moreover, neo-additive capacities allow for both ambiguity-averse and ambiguity-loving decision-makers.

[7] A capacity v on a set S is a set function $v : S \to [0,1]$ such that $v(\varnothing) = 0, v(S) = 1$, and $v(A) \geq v(B)$ for any $A, B \subseteq S$ and $B \subseteq A$.

[8] For further justification of this definition of support of a capacity, see Eichberger and Kelsey (2014).

[9] See Eichberger and Kelsey (2014) for a discussion of this issue.

[10] Most of the results in the present paper can be generalized to a setting where beliefs are in the class of JP-capacities (Jaffray and Philippe, 1997, Eichberger and Kelsey, 2014).

It is assumed that all participants in the contest have Choquet expected utility (CEU) preferences (Schmeidler, 1986, 1989) with player i's expected payoff function given by the Choquet integral of the payoff function in (2):[11]

$$W_i\left(x_i;\pi_i,\alpha_i,\delta_i\right)=\int U_i\left(x_i;\sum_{j\neq i}h_j\left(x_j\right)\right)dv_i\left(\mathbf{x}_{-i}\right). \tag{4}$$

Using the definition of a neo-additive capacity from (3), we can re-write (4) as

$$W_i\left(x_i;\pi_i,\alpha_i,\delta_i\right)=\delta_i\left(1-\alpha_i\right)M_i\left(x_i\right)+\delta_i\alpha_i m_i\left(x_i\right)+\left(1-\delta_i\right)\int U_i\left(x_i;\sum_{j\neq i}h_j\left(x_j\right)\right)d\pi_i\left(\mathbf{x}_{-i}\right), \tag{5}$$

where

$$\begin{aligned}
M_i\left(x_i\right) &\equiv \max_{\mathbf{x}_{-i}\in\mathbf{X}_{-i}} U_i\left(x_i;\sum_{j\neq i}h_j\left(x_j\right)\right)=\frac{h_i\left(x_i\right)}{h_i\left(x_i\right)+\underline{Y}_{-i}}V_i-x_i,\\
m_i\left(x_i\right) &\equiv \min_{\mathbf{x}_{-i}\in\mathbf{X}_{-i}} U_i\left(x_i;\sum_{j\neq i}h_j\left(x_j\right)\right)=\frac{h_i\left(x_i\right)}{h_i\left(x_i\right)+\bar{Y}_{-i}}V_i-x_i,\\
\underline{Y}_{-i} &\equiv \sum_{j\neq i}h_j\left(\underline{x}_j\right)\text{ and }\bar{Y}_{-i}\equiv\sum_{j\neq i}h_j\left(\bar{x}_j\right).
\end{aligned}$$

The function $M_i\left(x_i\right)$ represents the best possible scenario of player i's opponents' choices for player i while $m_i\left(x_i\right)$ corresponds to the worst possible scenario. These two functions will in general depend on player i's choice due to the strategic nature of the interaction between the contestants.

A neo-additive capacity has the following intuitive interpretation and behavioral implications. A decision-maker with CEU preferences and a neo-additive capacity has subjective beliefs characterized by the additive probability distribution $\pi_i\left(\cdot\right)$ but lacks confidence in this belief. When $\delta_i=0$, the decision-maker is certain in her probabilistic assessment $\pi_i\left(\cdot\right)$ and, as a result, has expected utility preferences. In contrast, when $\delta_i>0$, she/he will take into account the effect of her actions on the best and worst outcomes. The larger the parameter δ_i, the greater the weight that the decision-maker will place on these two extreme outcomes

[11] Chateauneuf, Eichberger, and Grant (2007) provide an axiomatic foundation of CEU preferences with a neo-additive capacity.

and the larger the deviation from the expected utility preferences. Thus, it is natural to interpret δ_i as measuring ambiguity, and we shall refer to it as the degree of ambiguity. The decision-maker's reaction to uncertainty about beliefs has optimistic and pessimistic traits. The optimistic trait is reflected by the weight on the best outcome $M_i(x_i)$, measured by $\delta_i(1-\alpha_i)$, while the pessimistic trait is given by the weight on the worst outcome $m_i(x_i)$, measured by $\delta_i\alpha_i$. Relatively high (low) values of α_i correspond to pessimistic (optimistic) attitudes to ambiguity. Thus, parameter α_i is referred to as the degree of pessimism (or degree of ambiguity aversion).

There is also a somewhat different interpretation of the payoff function in (5). Suppose that each of the players perceives uncertainty about her opponents' types. She entertains the possibility that, for each choice that she makes, her opponents will behave in a "collectively spiteful" fashion in the sense that they will play strategies that yield her the lowest possible payoff. The player also allows for the possibility that, for each choice that she makes, her opponents will behave in a "collectively altruistic" fashion in the sense that they will play strategies that yield her the highest possible payoff. Finally, the player does not discard the possibility that her opponents are "normal" in the sense that they will behave in a self-interested manner. Although this type of motivation of our analysis also seems plausible, in what follows, we focus on interpretations in terms of perceptions and attitudes to ambiguity.

Substituting from (2) into (5), we obtain:

$$\begin{aligned}
&W_i(x_i;\pi_i,\alpha_i,\delta_i)\\
= &\left[\delta_i\left(\frac{(1-\alpha_i)h_i(x_i)}{h_i(x_i)+\underline{Y}_{-i}}+\frac{\alpha_i h_i(x_i)}{h_i(x_i)+\bar{Y}_{-i}}\right)+(1-\delta_i)\int\left(\frac{h_i(x_i)}{h_i(x_i)+Y_{-i}}\right)d\pi_i(\mathbf{x}_{-i})\right]V_i-x_i,
\end{aligned}$$

where $Y_{-i}\equiv\sum_{j\neq i}h_j(x_j)$. Note that when $\underline{x}_1=\cdots=\underline{x}_n=0$ and $\bar{x}_1=\cdots=\bar{x}_n=\infty$, the payoff function can be written as

$$W_i(x_i;\pi_i,\alpha_i,\delta_i)=\left(\delta_i(1-\alpha_i)+(1-\delta_i)\int p_i(x_i;\mathbf{x}_{-i})\,d\pi_i(\mathbf{x}_{-i})\right)V_i-x_i.$$

Following Eichberger, Kelsey and Schipper (2009) and Eichberger and Kelsey (2014), we suppose that, given the structure of the game, $\pi_i(\cdot)$s are determined endogenously while the degrees of optimism, α_i, and ambiguity, δ_i, are treated as exogenous parameters. A decision-maker's attitude towards ambiguity α_i is a personal trait akin to tastes in a standard consumer problem. Thus, it is reasonable to suppose that it is independent of the decision problem and exogenous influences. To simplify our analysis, δ_i's are also treated as exogenous parameters.

Define the best-response correspondence of player i given that her/his beliefs are represented by a neo-additive capacity v_i by $R_i(v_i) = R_i(\pi_i, \alpha_i, \delta_i) \equiv \underset{x_i \in X_i}{argmax} W_i(x_i; \pi_i, \alpha_i, \delta_i)$. We adopt the following definition from Eichberger and Kelsey (2014), which is an extension of the definition put forth by Dow and Werlang (1994):

Definition 1 *(equilibrium under ambiguity) A vector of neo-additive capacities* $(\hat{v}_1, \hat{v}_2, ..., \hat{v}_n)$ *is an Equilibrium Under Ambiguity (EUA) if for all players* $i = 1, 2, .., n$, $\varnothing \neq$ *supp* $\hat{v}_i \subseteq \underset{j \neq i}{\times} R_j(\hat{v}_j)$. *If* $\hat{\mathbf{x}}_{-i} \in$ *supp* $\hat{v}_i$ *for all* $i = 1, 2, .., n$, *then* $(\hat{x}_1, \hat{x}_2, ..., \hat{x}_n)$ *is called an equilibrium strategy profile. If supp* $\hat{v}_i$ *contains a single vector* $\hat{\mathbf{x}}_{-i}$ *for each player* $i = 1, 2, .., n$, *we will say that* $\hat{\mathbf{x}}$ *is a singleton equilibrium.*

Thus, an equilibrium is characterized by a capacity for each player. The support of this capacity consists of strategies that are best responses for the opponents.

Consider the game

$$\Gamma(\boldsymbol{\delta}, \boldsymbol{\alpha}) = \left\langle \left(\mathbf{X}_i, \delta_i (1 - \alpha_i) M_i(x_i) + \delta_i \alpha_i m_i(x_i) + (1 - \delta_i) U_i(x_i; \mathbf{x}_{-i})\right)_{i=1,2,...,n} \right\rangle,$$

where $\boldsymbol{\delta} \equiv (\delta_1, ..., \delta_n)$ and $\boldsymbol{\alpha} \equiv (\alpha_1, ..., \alpha_n)$. Note that $\Gamma(\boldsymbol{\delta}, \boldsymbol{\alpha})$ is a 'perturbed' game obtained from $G = \left\langle \left(\mathbf{X}_i, U_i(x_i; \mathbf{x}_{-i})\right)_{i=1,2,...,n} \right\rangle$ by replacing $U_i(x_i; \mathbf{x}_{-i})$ with the function $\delta_i (1 - \alpha_i) M_i(x_i) + \delta_i \alpha_i m_i(x_i) + (1 - \delta_i) U_i(x_i; \mathbf{x}_{-i})$ for $i = 1, 2, ..., n$. Eichberger, Kelsey and Schipper (2009) establish a relationship between the sets of Nash equilibria of the perturbed game and Equilibria under Ambiguity for games with two players. Their arguments can be extended to games with an arbitrary number of players to obtain the following:

Lemma 2 *For any pure strategy Nash equilibrium* $(x_1^*, ..., x_n^*)$ *of the perturbed game* $\Gamma(\boldsymbol{\delta}, \boldsymbol{\alpha})$, *there is a corresponding singleton equilibrium under ambiguity* $(v_1^*, ..., v_n^*)$ *of the game* G *with* $v_i^* = \delta_i (1 - \alpha_i) + (1 - \delta_i) \pi_i^*$ *and* $\pi_i^* \left(\mathbf{x}_{-i}^*\right) = 1$ *for* $i = 1, ..., n$.

In light of the established relationship between the equilibria under ambiguity of the game G and Nash equilibria of the perturbed game, we examine the pure strategy equilibria of the latter game. Since the payoff function of the perturbed game can be written as

$$Z_i(x_i, \sum_{j \neq i} h_j(x_j), \alpha_i, \delta_i) = \left[\delta_i\left((1-\alpha_i)\, p_i\left(x_i; \underline{\mathbf{x}}_{-i}\right) + \alpha_i p_i\left(x_i; \bar{\mathbf{x}}_{-i}\right)\right) + (1-\delta_i)\, p_i\left(x_i; \mathbf{x}_{-i}\right)\right] V_i - x_i,$$

the perturbed game represents a contest where player i's probability of winning the prize is equal to

$$\delta_i\left((1-\alpha_i)\, p_i\left(x_i; \underline{\mathbf{x}}_{-i}\right) + \alpha_i p_i\left(x_i; \bar{\mathbf{x}}_{-i}\right)\right) + (1-\delta_i)\, p_i\left(x_i; \mathbf{x}_{-i}\right)$$

and the value of the prize is equal to V_i. The latter expression reveals that the incentives to invest in the contest come through three different channels; the optimistic scenario $\delta_i(1-\alpha_i)\, p_i\left(x_i; \underline{\mathbf{x}}_{-i}\right)$, the pessimistic scenario $\delta_i \alpha_i p_i\left(x_i; \bar{\mathbf{x}}_{-i}\right)$, and the "standard" scenario $(1-\delta_i)\, p_i\left(x_i; \mathbf{x}_{-i}\right)$.

Player i's best response function[12] for $\sum_{j \neq i} h_j(x_j) > 0$ is given by:

$$\phi_i\left(\sum_{j \neq i} h_j(x_j)\right) = \begin{cases} \underline{x}_i, & \text{if } \frac{\partial Z_i\left(\underline{x}_i; \sum_{j \neq i} h_j(x_j)\right)}{\partial x_i} \leq 0 \\ \bar{x}_i, & \text{if } \frac{\partial Z_i\left(\bar{x}_i; \sum_{j \neq i} h_j(x_j)\right)}{\partial x_i} > 0 \\ \text{unique positive solution of } \frac{\partial Z_i\left(x_i, \sum_{j \neq i} h_j(x_j), \alpha_i, \delta_i\right)}{\partial x_i} = 0, & \text{otherwise} \end{cases}. \tag{6}$$

The form of the best response function in (6) follows from the strict concavity of $p_i(x_i; \mathbf{x}_{-i})$ in x_i for all $x_i > 0$ and all $\mathbf{x}_{-i}$ and the resultant strict concavity of the objective function in (6). We also have that player i's best response function is continuous for $\sum_{j \neq i} h_j(x_j) > 0$. In the Appendix we prove that the examined strategic interaction has a singleton equilibrium under ambiguity. The proof uses Lemma 2 and Reny's (1999) conditions for the existence of a pure-strategy Nash equilibrium:

Proposition 3 *The game G has a singleton equilibrium under ambiguity $(v_1^*, ..., v_n^*)$ where $v_i^* = \delta_i(1-\alpha_i) + (1-\delta_i)\pi_i^*$, $\pi_i^*\left(\mathbf{x}_{-i}^*\right) = 1$ for $i = 1, ..., n$ and $\mathbf{x}^*$ is a pure strategy equilibrium of the perturbed game given by $\left(\phi_1\left(\sum_{j \neq 1} h_j\left(x_j^*\right)\right), \phi_2\left(\sum_{j \neq 2} h_j\left(x_j^*\right)\right), ..., \phi_n\left(\sum_{j \neq n} h_j\left(x_j^*\right)\right)\right)$.*

[12] Acemoglu and Jensen (2013) call it a "reduced" best-reply function.

Proof. See Appendix. ■

This singleton equilibrium is the ambiguous equivalent of a pure strategy Nash equilibrium. There may be other equilibria under ambiguity, in which there are two or more strategies in the support of the players' beliefs. These correspond to mixed strategy Nash equilibria. We focus on the singleton equilibrium since, even in the absence of ambiguity, the interpretation of mixed equilibrium is problematic. Since players are indifferent between all of the strategies to which they assign a positive probability they have no incentive to play the strategy which sustains the mixed equilibrium (see, e.g., Osborne and Rubinstein, 1994). In the presence of ambiguity the issues are even more complicated since one would need to determine what it means for a player to use an ambiguous randomizing device to select his strategy. However, given that (the analogies of) pure strategy equilibria always exist in our model, it is desirable to avoid these issues by confining attention to such equilibria.

3 Symmetric case

We begin with an analysis[13,14] of a symmetric contest where all of the players have the same value of the prize, the same contest success function, the same lower and upper bounds on contest functions, and the same degrees of pessimism:

$$\begin{aligned} V_1 &= \cdots = V_n \equiv V,\ h_1(\cdot) = \cdots = h_n(\cdot) = h(\cdot), \\ \underline{x}_1 &= \cdots = \underline{x}_n \equiv \underline{x},\ \bar{x}_1 = \cdots = \bar{x}_n \equiv \bar{x}, \\ \alpha_1 &= \cdots = \alpha_n \equiv \alpha,\ \delta_1 = \cdots = \delta_n \equiv \delta. \end{aligned}$$

Suppose also that $h(x) = x^{\beta}$, where $\beta \leq 1$ (Tullock, 1967, 1980). There are a number of reasons we examine symmetric contests. First, they are more tractable. Second, they

[13] We only focus on the comparative statics for the parameters that distinguish our framework from the received literature, namely, those that are associated with ambiguity about opponents' behavior. The comparative statics findings for other parameters, such as the value of the prize to a player or the number of contenders in a contest, are offered by the existing literature (e.g., Nti, 1997, Yamazaki, 2008, and Acemoglu and Jensen, 2013).

[14] Note that the games considered in the present paper fall into the category that Acemoglu and Jensen (2013) coin "nice" aggregative games. However, Acemoglu and Jensen's (2013) results are not applicable to the parameters of interest in our model, since neither α nor δ are, what they call, "positive shocks" for our payoff functions.

are more illustrative of how the degree of ambiguity and degree of ambiguity aversion affect behavior in contests. Third, many experimental studies entail various symmetry assumptions and we are interested in juxtaposing our findings to the received experimental evidence.[15]

In a symmetric equilibrium, $x_1^* = ... = x_n^* \equiv x^*$. From the first-order conditions in (6), we obtain an implicit expression for the unique interior symmetric equilibrium (when it exists)[16]:

$$F(x^*, \alpha, \delta) \equiv \beta (n-1) (x^*)^{\beta-1} \left[\delta \left(\begin{array}{c} (1-\alpha) \frac{\underline{x}^\beta}{\left((x^*)^\beta + (n-1)\underline{x}^\beta\right)^2} \\ +\alpha \frac{\bar{x}^\beta}{\left((x^*)^\beta + (n-1)\bar{x}^\beta\right)^2} \end{array} \right) + (1-\delta) \frac{1}{n^2 (x^*)^\beta} \right] V - 1 = 0. \tag{7}$$

3.1 Degree of ambiguity and equilibrium effort

The effect of the degrees of ambiguity δ on the equilibrium effort is characterized in the following:

Proposition 4 *The equilibrium effort x^* under ambiguity will decrease in the degree of ambiguity δ if and only if the equilibrium effort $x^N \equiv \frac{\beta(n-1)}{n^2} V$ without ambiguity exceeds the equilibrium effort x^* under ambiguity:*

$$x^N \geq x^*. \tag{8}$$

Proof. See Appendix. ■

In the Appendix we also demonstrate that condition (8) holds if and only if

$$\frac{(1-\alpha)\underline{x}^\beta}{\left((\beta(n-1)V)^\beta + n^{2\beta}(n-1)\underline{x}^\beta\right)^2} + \frac{\alpha \bar{x}^\beta}{\left((\beta(n-1)V)^\beta + n^{2\beta}(n-1)\bar{x}^\beta\right)^2} < \frac{1}{n^{2\beta+2}(\beta(n-1)V)^\beta}, \tag{9}$$

[15] It is of course very unlikely that all of the participants in a contest will perceive the same degree of ambiguity and have the same degree of ambiguity aversion. The assumption that all players have the same perception of ambiguity and attitude to ambiguity is a simplifying assumption. We relax this assumption in later sections of the paper.

[16] Existence and uniqueness of an interior symmetric equilibrium is guaranteed by imposing restrictions on the lower and upper bounds for x and other parameters of the model. The comparative statics analysis is straightforward when either all of the players choose the lower bound on expenditure or all of the players choose the upper bound. For this reason, we focus on the interior solutions in the symmetric case.

which reveals how the model parameters affect the relationship between the degree of ambiguity δ and the equilibrium effort x^*. Under both inequality (8) and its reverse, an increase in ambiguity widens the gap between the equilibrium effort under ambiguity and the Nash prediction. If the ambiguity attitude of the contenders (and other parameters of the model) is such that the equilibrium effort under ambiguity exceeds the Nash prediction x^N, then an increase in the degree of ambiguity will widen this gap by increasing the equilibrium effort under ambiguity. Figure 1 depicts this scenario for a parameterization of our model where the contenders are ambiguity seeking ($\alpha = 0.3$). The Nash equilibrium for this example is equal to 1.6 and the equilibrium effort under ambiguity continuously increases starting from this level as the degree of ambiguity changes from no ambiguity ($\delta = 0$) to total ambiguity ($\delta = 1$).

When $x^N \geq x^*$, an increase in ambiguity widens the gap by decreasing the equilibrium effort under ambiguity. Figure 2 depicts this possibility for the case of ambiguity averse contenders ($\alpha = 0.9$). Note also that the contenders may be ambiguity loving but underinvest compared to the Nash equilibrium. It is easy to construct examples where the participants are ambiguity seeking but where x^* is decreasing in δ. More generally, it follows from inequality (8) that if

$$\frac{\underline{x}^{\beta}}{\left(\left(\beta\left(n-1\right)V\right)^{\beta}+n^{2\beta}\left(n-1\right)\underline{x}^{\beta}\right)^{2}} > \frac{\bar{x}^{\beta}}{\left(\left(\beta\left(n-1\right)V\right)^{\beta}+n^{2\beta}\left(n-1\right)\bar{x}^{\beta}\right)^{2}} \tag{10}$$

then there is a threshold level of the degree of ambiguity aversion such that x^* is increasing in δ if and only if the participants have a degree of ambiguity aversion that is smaller than that threshold level. Conversely, if the reverse of inequality (10) holds then x^* is increasing in δ if and only if the participants have a degree of ambiguity aversion that is higher than some threshold level of ambiguity aversion. As another illustration of Proposition 4, consider the case where there is no upper bound on the expenditure ($\bar{x} = \infty$) and $\underline{x} > 0$. Inequality (10) holds in this case for all values of the model parameters and, hence, x^* is increasing in δ if and only if the participants are sufficiently ambiguity seeking.

It also follows from condition (9) that the equilibrium effort x^* either increases in the degree of ambiguity for all δ or decreases in the degree of ambiguity for all δ. Equivalently, whether the contenders underinvest or overinvest relative to the Nash equilibrium is independent of the degree of ambiguity $\delta \in (0,1)$.

Finally, consider the case where $0 < \bar{x} < \infty$ and $\beta = 1$. This is a simple lottery frequently explored in experimental studies. The parameter $\bar{x}$ may represent the subjects' endowment of experimental currency. Suppose also that the participants in the lottery believe that their opponents will buy at least $\underline{x} > 0$ lottery tickets. Under this scenario, a contender's equilibrium effort under ambiguity will exceed the Nash equilibrium if and only if

$$(1-\alpha)\frac{\underline{x}}{(n-1)\left(V+n^2\underline{x}\right)^2}+\alpha\frac{\bar{x}}{(n-1)\left(V+n^2\bar{x}\right)^2}-\frac{1}{n^4V}>0.$$

Figure 3 depicts the left-hand-side of the above inequality as a function of the representative contender's pessimism keeping the values of the other parameters fixed. It reveals that the equilibrium effort under ambiguity exceeds the Nash equilibrium if and only if the contender is sufficiently optimistic. The threshold value of the degree of pessimism is about 0.38 in this example. Note also that this is illustrative of a more general pattern for symmetric contests. Namely, if the upper threshold $\bar{x}$ is relatively large compared to the lower threshold $\underline{x}$, then the equilibrium effort under ambiguity will exceed the Nash equilibrium if and only if the contenders are sufficiently optimistic.

3.2 Degree of pessimism and equilibrium effort

The relationship between the equilibrium effort and the contenders' pessimism is given by:

Proposition 5 *The equilibrium effort x^* under ambiguity will increase in the degree of pessimism α if and only if*

$$x^* > (n-1)^{\frac{1}{\beta}}\sqrt{\underline{x}\bar{x}}, \tag{11}$$

which holds if and only if

$$\frac{\delta}{\left(\sqrt{\underline{x}\bar{x}}\right)^{1-\beta}\left(\bar{x}^{\frac{\beta}{2}}+\underline{x}^{\frac{\beta}{2}}\right)^2}+\frac{(1-\delta)(n-1)}{n^2\left(\sqrt{\underline{x}\bar{x}}\right)}>\frac{(n-1)^{\frac{1}{\beta}}}{\beta V}. \tag{12}$$

Proof. See Appendix. ■

It follows immediately from (12) that:

Corollary 6 *The equilibrium effort x^* under ambiguity will decrease in the degree of pessimism α if and only if*
(i) the value of the prize V is sufficiently small, and/or
(ii) the number of contestants n is sufficiently large, and/or
(iii) the lower bound $\underline{x}$ on effort is sufficiently large, and/or
(iv) the upper bound $\bar{x}$ on effort is sufficiently large.

To gain intuition into the necessary and sufficient conditions for the equilibrium effort to be decreasing in the degree of pessimism, recall the three channels through which changes in the parameters affect the incentives to invest in the contest; the optimistic, pessimistic, and standard channels. Consider a decrease in the value of the prize V. The incentives to invest for all three channels will decrease. Moreover, the disincentives to invest associated with the pessimistic channel will be more prominent for a lower value of the prize. Hence, the equilibrium effort will be lower for contests with relatively pessimistic contenders and low value of the prize. Similar reasoning underlies part (ii) of Corollary 6. Consider now part (iv) of the Corollary (part (iii) has a similar intuition). A relatively pessimistic contender places most of the weight on the scenario where her opponents choose a relatively large expenditure, namely the upper bound $\bar{x}$. Part (iv) of the Corollary follows because an increase in $\bar{x}$ results in a decrease in the marginal benefit of own action and because this effect is stronger when the contender is relatively pessimistic. Finally, note that when there is no upper bound on the expenditure ($\bar{x} = \infty$) and the lower bound is strictly positive ($\underline{x} > 0$), the equilibrium effort x^* is decreasing in the degree of pessimism α.

4 The model with two types of contenders

Suppose as in the previous section that:

$$V_1 = \cdots = V_n \equiv V,\ \underline{x}_1 = \cdots = \underline{x}_n \equiv \underline{x},\ \bar{x}_1 = \cdots = \bar{x}_n \equiv \bar{x}.$$

But, in contrast to the preceding section, the contenders may differ in terms of their degrees of ambiguity δ and their degrees of ambiguity aversion α. There are two types of contenders; type-A contenders, that have a common degree of ambiguity δ_A and common degree of ambiguity aversion α_A, and type-B contenders, that have a common degree of ambiguity δ_B and common degree of ambiguity aversion α_B. The contenders $\{1, ..., m\}$ (for $1 \leq m \leq n-1$) are of type A while the remaining contenders are of type B. For simplicity, we assume that $h_1(\cdot) = \cdots = h_n(\cdot) = h(x) = x$, thus, restricting the setup in this section to simple lotteries. This assumption can be relaxed but at the expense of substantially cluttering the exposition.

We focus on an equilibrium where all contenders of the same type choose the same action; $x_1^* = ... = x_m^* \equiv x_A^*$ and $x_{m+1}^* = ... = x_n^* \equiv x_B^*$. From the first-order conditions in (6), the equilibrium actions (x_A^*, x_B^*) for an interior equilibrium are implicitly given by the following system of equations[17]:

$$(n-1)\,\delta_A\left[\frac{(1-\alpha_A)\,\underline{x}}{(x_A+(n-1)\,\underline{x})^2}+\frac{\alpha_A\bar{x}}{(x_A+(n-1)\,\bar{x})^2}\right]+(1-\delta_A)\,\frac{(m-1)\,x_A+(n-m)\,x_B}{(mx_A+(n-m)\,x_B)^2}=\frac{1}{V},$$

$$(n-1)\,\delta_B\left[\frac{(1-\alpha_B)\,\underline{x}}{(x_B+(n-1)\,\underline{x})^2}+\frac{\alpha_B\bar{x}}{(x_B+(n-1)\,\bar{x})^2}\right]+(1-\delta_B)\,\frac{mx_A+(n-m-1)\,x_B}{(mx_A+(n-m)\,x_B)^2}=\frac{1}{V}. \tag{13}$$

4.1 The model with two contenders

For compactness of our presentation, a large part of the following analysis is conducted for the case where there is one player of each type, players A and B. This assumption is without substantial loss of generality since the strategic interaction in the general case is a game with aggregate externalities so that each player is playing against the aggregate of all the other players. To further reduce the number of possible cases and to avoid some uninteresting solutions it is assumed that $\bar{x} > \frac{V}{4} > \underline{x} > 0$.

In a contest with two contenders, player A's and B's best response functions, $\phi_A(x_B)$ and $\phi_B(x_A)$, for interior solutions are given by the unique solutions to the corresponding equations in (13) with $n = 2$ and $m = 1$. They satisfy the following property:

[17] As in the previous section, an interior equilibrium materializes for certain conditions on the lower and upper bounds for x and other parameters of the model. See the following sections for analysis of corner solutions.

Lemma 7 *The slopes of the reaction functions at interior points ($\underline{x} < x_i, x_j < \bar{x}$) satisfy*

$$\frac{\partial \phi_i}{\partial x_j} > 0 \text{ if } x_i > x_j \text{ and } \frac{\partial \phi_i}{\partial x_j} < 0 \text{ if } x_i < x_j \text{ where } i, j \in \{A, B\} \text{ and } i \neq j.$$

Proof. See Appendix. ■

For $x_i > x_j$ a marginal increase in the opponent's action x_j intensifies the competition leading to an increase in player's i's effort. In contrast, for $x_j > x_i$ an increase in x_j reduces the intensity of the competition leading to a decrease in player's i's effort. Figures 4-6 depict the two players' best response curves for various constellations of model parameters. In each of these figures, player A's (player B's) best response curve is downward (upward) sloping above the 45^0 degree line and upward (downward) sloping below it.

As a benchmark, we use the scenario where neither player perceives ambiguity about the opponent's strategy ($\delta_A = \delta_B = 0$). The unique Nash equilibrium in this case is given by $x_A = x_B = \frac{V}{4}$. Its relationship to the equilibrium under ambiguity in a two-player contest is characterized in the following:

Proposition 8 *Suppose that both players perceive a positive degree of ambiguity ($\delta_A, \delta_B > 0$). Then, in an equilibrium under ambiguity both players will make strictly less than the Nash equilibrium level of contributions.*

Proof. See Appendix. ■

The intuition behind this finding is as follows. Recall that the incentives to invest come through three channels. The pessimistic and optimistic channels produce incentives to invest that are lower than the incentives for the game without ambiguity. A complete pessimist believes that her opponent will choose a very large investment. In this case, the marginal product of the player's effort is relatively low since she believes that she will likely lose the contest unless she invests a very large amount. The marginal product of effort is also relatively low for an optimist since there is only one opponent[18] and optimism causes a player to overweight low effort from her opponent. In this case, the player believes that she can win the contest without much effort. Ambiguity causes the decision-maker to place weights

[18]With multiple opponents, the pessimistic channel may induce more effort from a player than under Nash.

on both possibilities. As a result, the equilibrium expenditures under ambiguity are lower than under the Nash equilibrium.

We structure the rest of this section by first analyzing how changes in the players' common and individual degrees of ambiguity and attitudes to ambiguity affect equilibrium efforts starting from their levels under a symmetric contest. We then report comparative statics results for three asymmetric specifications of degrees of ambiguity and attitudes to ambiguity in a two-player contest with $\bar{x} = V$. Subsequently, we consider the more general case when there are multiple players of each of the two types and examine how the number of players of each type affects equilibrium behavior.

4.1.1 Changes to a symmetric contest

As a starting point in the comparison of equilibrium efforts we take a two-player symmetric contest where both players perceive the same degree of ambiguity δ and have the same degree of pessimism α. For such contests, we have:

Proposition 9 *Suppose $\frac{1-\delta\alpha}{4\underline{x}} + \frac{\delta\alpha\bar{x}}{(\underline{x}+\bar{x})^2} > \frac{1}{V}$. Then, the two-player symmetric contest has a unique equilibrium. Moreover, this equilibrium is symmetric and interior.*

Proof. See Appendix. ■

When $\frac{1-\delta\alpha}{4\underline{x}} + \frac{\delta\alpha\bar{x}}{(\underline{x}+\bar{x})^2} \leq \frac{1}{V}$, both players will choose the lowest possible expenditure $\underline{x}$. We sidestep this uninteresting case and instead focus on interior solutions in the rest of this section. It follows immediately from Propositions 4 and 8 that for a symmetric contest with two contenders, the unique equilibrium effort is a strictly decreasing function of the common degree of ambiguity δ. Proposition 5 in turn implies that for a symmetric two-contender contest the symmetric equilibrium will be a decreasing function of the common degree of pessimism if and only if

$$x^* < \sqrt{\underline{x}\bar{x}},$$

which implies by Proposition 8 that if $\underline{x}\bar{x} > \frac{V^2}{16}$ then the symmetric equilibrium will be a decreasing function of the degree of pessimism. The effect of an increase in the degree of pessimism is to shift the decision weight from the best outcome to the worst outcome. Under inequality $\underline{x}\bar{x} > \frac{V^2}{16}$, the extra weight on the worst outcome affects the marginal benefit more than the reduction of the weight on the best outcome.

The best response functions in a symmetric contest satisfy:

Lemma 10 *Let $i, j \in \{A, B\}$ and $i \neq j$. We have*

1. $\phi_i(x_j) \gtreqless x_j$ as $x_j \lesseqgtr x^*$,

2. $\phi_i(x_j)$ is increasing (resp. decreasing) on $[\underline{x}, x^*]$, (resp. $[x^*, \bar{x}]$).

Proof. See Appendix. ■

Thus, the best response functions are single-peaked with the peak located at the unique symmetric equilibrium (see the following proposition). Note also that monotonicity of the best response functions is strict for interior solutions (see Figure 4).

Using Lemma 10, we can determine how changes in the perceptions of ambiguity and attitudes to ambiguity of individual players affect behavior starting from the symmetric environment. Let $x_A = x_B = x$ denote the equilibrium effort level for a symmetric contest with $\delta_A = \delta_B = \delta$ and $\alpha_A = \alpha_B = \alpha$ and let (x'_A, x'_B) denote the equilibrium for an asymmetric contest with $\delta_j = \delta < \delta_i = \delta'$ and $\alpha_A = \alpha_B = \alpha$, where $i, j \in \{A, B\}$ and $i \neq j$.[19] Figure 4 depicts the effect of an increase in player A's degree of ambiguity on equilibrium behavior starting from the symmetric scenario. As a result of this change, player A's best response curve shifts leftward while player B's best response curve remains unchanged. The equilibrium under ambiguity moves from point A to point A'. Formally, we have:

Proposition 11 *An increase in player i's ($i \in \{A, B\}$) perception of ambiguity starting from a symmetric contest will strictly decrease both players' equilibrium efforts; $x'_k < x_k$ for $k \in \{A, B\}$. Moreover, the resulting reduction in player i's effort will strictly exceed the reduction in player j's ($j \neq i$) effort; $x'_j > x'_i$.*

Proof. See Appendix. ■

An increase in ambiguity perceived by Player i causes her to put more weight on the possibility that her opponent will choose very high expenditure $\bar{x}$ or very low expenditure $\underline{x}$.

[19] It is implicitly assumed that an increase in the degree of ambiguity from δ to δ' is relatively small. See the proof of the following proposition for more details.

This decreases player i's perceived marginal benefit and, as a result, reduces her equilibrium effort. Since the competition from player i has become less intense, player $j \neq i$ responds by decreasing her effort as well. However, to stay ahead of her opponent in terms of having a higher probability of winning, player j reduces effort by less than player i. Thus, an increase in player i's ambiguity perception renders strategic advantage to player j and improves the latter player's payoff. Proposition 11 also implies that when the two contenders are involved in a rent-seeking activity (Tullock, 1967), an increase in ambiguity perceived by either player will decrease the amount of rent dissipation.

Consider the effect of changes in player i's degree of ambiguity aversion and now let (x'_A, x'_B) denote the equilibrium for an asymmetric contest with $\delta_A = \delta_B = \delta$ and $\alpha_j = \alpha < \alpha_i = \alpha'$, where $i, j \in \{A, B\}$ and $i \neq j$. Conducting analysis similar to that for the previous proposition, we obtain:

Proposition 12 *An increase in player i's ($i \in \{A, B\}$) perception of ambiguity aversion starting from a symmetric contest will decrease both players' equilibrium efforts; $x'_k < x_k$ for $k \in \{A, B\}$. Moreover, the resulting reduction in player i's effort will exceed the reduction in player j's ($j \neq i$) effort; $x'_j > x'_i$.*

In the following sections we consider scenarios where the players' initial perceptions of ambiguity and their attitudes to ambiguity can be asymmetric.

4.1.2 Both players are pessimists: $\alpha_A = \alpha_B = 1$

Using arguments similar to those for Propositions 11 and 12, we can show that changes in the players' degrees of ambiguity have the following effects on the equilibrium expenditures[20]:

Proposition 13 *Suppose that initially players A and B have degrees of ambiguity $\delta_i = \tilde{\delta} < \delta_j = \hat{\delta}$, where $i \in \{A, B\}$ and $j \neq i$. Then:*

1. *An increase in δ_i from $\tilde{\delta}$ to δ', where $\tilde{\delta} < \delta' \leqslant \hat{\delta}$, will lead to a decrease in x_i^* and an increase in x_j^*.*

2. *A decrease in δ_i will lead to an increase in x_i^* and a decrease in x_j^*.*

[20] As for Propositions 11 and 12, we are implicitly assuming that these changes in the degrees of ambiguity are sufficiently small.

3. An increase in δ_j will lead to decreases in both x_i^* and x_j^*.

4. A decrease in δ_j from $\hat{\delta}$ to δ'', where $\tilde{\delta} < \delta'' \leqslant \hat{\delta}$, will lead to increases in both x_i^* and x_j^*.

To gain intuition into Proposition 13, suppose that $i = B$ and $j = A$. The initial equilibrium under ambiguity will correspond to a point like A' in Figure 4. Suppose, for example, that player B's perception of ambiguity increases. As a result, player B's best response curve will shift downward while player A's best response curve will remain unchanged. The new equilibrium will entail a lower expenditure by player B and higher expenditure by player A (part 1 of Proposition 13).

4.1.3 Player A is a pessimist, player B is ambiguity neutral: $\alpha_A = 1$, $\delta_B = 1$

Suppose that player A is a complete pessimist ($\alpha_A = 1$) while player B does not perceive any ambiguity ($\delta_B = 1$). There are two qualitatively different types of equilibria under ambiguity. When player A perceives relatively little ambiguity, both players choose expenditures that are strictly above the lower threshold $\underline{x}$ and below the Nash equilibrium level of $\frac{V}{4}$. This case is depicted in Figure 5 where point A represents the equilibrium under ambiguity. Note that the ambiguity neutral player invests more than the ambiguity averse player and has a higher chance of winning the prize under the equilibrium under ambiguity. In contrast, under the Nash equilibrium (point N in Figure 5) both players invest the same amount and have an equal chance of winning the prize.

The second scenario arises when player A's beliefs are relatively ambiguous. This case is depicted in Figure 6 where the equilibrium under ambiguity, represented by point A, entails the lowest possible level of investment $\underline{x}$ by player A and a strictly higher investment $\sqrt{V\underline{x}} - \underline{x}$ by player B. Thus, similarly to the first scenario, player B has a higher chance of winning the prize than player A. Formally, we have the following:

Proposition 14 *1. If $\delta_A \leq \hat{\delta}_A \equiv \frac{(\sqrt{V} - 2\sqrt{\underline{x}})(\underline{x}+V)^2}{(\sqrt{V} - \sqrt{\underline{x}})(\underline{x}+V)^2 - V^2\sqrt{\underline{x}}}$ then the game has a unique equilibrium under ambiguity (x_A^*, x_B^*) which is implicitly given by*

$$\delta_A \frac{V}{(x_A^* + V)^2} + (1 - \delta_A) \frac{\sqrt{Vx_A^*} - x_A^*}{Vx_A^*} = \frac{1}{V} \text{ and } x_B^* = \sqrt{Vx_A^*} - x_A^*. \tag{14}$$

2. If $\delta_A \geq \hat{\delta}_A$ then the game has a unique equilibrium under ambiguity (x_A^*, x_B^*) which is given by

$$x_A^* = \underline{x} \text{ and } x_B^* = \sqrt{V\underline{x}} - \underline{x}. \tag{15}$$

Proof. See Appendix. ■

We conclude the analysis in this section with an examination of the relationship between the equilibrium under ambiguity and player A's degree of ambiguity. Note that if player A's degree of ambiguity decreases marginally in the range characterized in part 2 of Proposition 14 then an increase in δ_A will leave the equilibrium under ambiguity intact. Thus, we focus on the case of an interior equilibrium under ambiguity from part 1 of Proposition 14.

Proposition 15 *Suppose $\delta_A < \hat{\delta}_A$. An increase in player A's degree of ambiguity δ_A will lead to a decrease in the equilibrium expenditure of both players.*

Proof. See Appendix. ■

As the degree of ambiguity increases from $\delta_A = 0$ to $\delta_A = \hat{\delta}_A$, Player A's expenditure monotonically decreases from the Nash equilibrium level $\frac{V}{4}$ to $\underline{x}$ while player B's expenditure monotonically decreases from $\frac{V}{4}$ to $\sqrt{V\underline{x}} - \underline{x}$. An increase in the degree of ambiguity causes player A to put more weight on the extreme outcome that player B will choose the highest possible expenditure V. Since the marginal benefit of expenditure is relatively small under this extreme scenario, player A's best response curve will exhibit a partial leftward shift as a result of an increase in δ_A.[21] In contrast, player B's best response curve will remain unchanged. Hence, the equilibrium under ambiguity will move along player B's best response curve to the point of intersection of player A's new best response curve and player B's unaffected best response curve. This point of intersection entails lower levels of expenditure by both players.

4.1.4 Player A is a pessimist, player B is an optimist: $\alpha_A = 1$, $\alpha_B = 0$

When player A is a complete pessimist ($\alpha_A = 1$) while player B is a complete optimist ($\alpha_B = 0$), we have:[22]

[21] The shift is partial because the vertical part of player B's best response curve does not shift left but shifts down.

[22] Due to space considerations, we relate the results on the effect of the degrees of ambiguity to the relationship between endogenous variables x_A^* and x_B^* rather than the relationship between the model para-

Proposition 16 *1. An increase in player A's degree of ambiguity δ_A will lead to*

(1i) *a decrease in the equilibrium expenditures of both players if $x_B^* > x_A^*$;*

(1ii) *a decrease in x_A^* and an increase in x_B^* if $x_B^* < x_A^* < \sqrt{Vx_B^*}$;*

(1iii) *an increase in x_A^* and a decrease in x_B^* if $x_A^* > \sqrt{Vx_B^*}$.*

2. An increase in player B's degree of ambiguity δ_B will lead to

(2i) *an increase in x_A^* and a decrease in x_B^* if $x_B^* > x_A^*$;*

(2ii) *a decrease in the equilibrium expenditures of both players if $\sqrt{x_A^* \underline{x}} < x_B^* < x_A^*$;*

(2iii) *an increase in the equilibrium expenditures of both players if $x_B^* < \sqrt{x_A^* \underline{x}}$.*

Proof. See Appendix. ■

Consider, for example, the case where the best-response curves intersect over the 45^0 degree line ($x_B^* > x_A^*$). In this case, an increase in the ambiguity averse player's degree of ambiguity results in a decrease of both players' expenditures. In contrast, an increase in the ambiguity loving player's degree of ambiguity results in an increase in the ambiguity averse player's expenditure and a decrease in the ambiguity loving player's expenditure.

4.2 Proportion of ambiguity averse players and equilibrium effort

We now return to the general case and suppose that there are m identical ambiguity averse contenders and $(n - m)$ identical ambiguity loving contenders. Figure 7 depicts the relationship between the number m of ambiguity averse contenders and the equilibrium expenditures of representative ambiguity averse and ambiguity loving contenders for a specific parameterization of out model. In this case, irrespective of the fraction of ambiguity averse players the equilibrium expenditure of ambiguity averse contenders is below the Nash equilibrium level (which is equal to 8) while the equilibrium expenditure of ambiguity loving contenders is above it. Note also that as the number of ambiguity averse players monotonically increases, both contender types increase their equilibrium expenditures.

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meters.

5 Conclusion

The paper developed and analyzed contests where contenders perceive ambiguity about strategies of their opponents. In addition to proving existence of equilibrium under ambiguity and exploring its uniqueness properties, we have investigated how the degree of ambiguity regarding other participants' strategies and preferences toward ambiguity affect equilibrium behavior. The paper also established a relationship between the equilibrium under ambiguity and Nash equilibrium.

Our results suggest that relatively optimistic players tend to invest more than their pessimistic counterparts. Pessimists place a relatively large weight on the event that their opponents will incur relatively large expenditures which in turn mutes incentives to expend resources. In contrast, optimists place most of the weight on the scenario that opponents will choose relatively small expenditures. The incentives to invest are stronger when opponents choose relatively small expenditures than under the most pessimistic scenario. As a result, optimists invest more and have a higher chance of winning the prize. This effect is especially pronounced when the number of opponents is relatively large.

The model yields a number of testable hypotheses. It would be informative to empirically investigate these in the lab and in the field. To accomplish this objective, one could use a two-stage procedure where in the first stage the subjects' attitudes to ambiguity are elicited using an Ellsberg style experimental design while in the second stage these subjects strategically interact in a contest. We leave this interesting experimental exercise to future research.

6 Appendix

Our proof of Proposition 3 utilizes the sufficient conditions for the existence of a pure strategy Nash equilibrium in Reny (1999). The analysis relies on the following definitions (see Reny (1999) for more details). A game is called compact if each player's pure strategy set is nonempty and compact and each player has a bounded payoff function. A pair $(\mathbf{x}^*, \mathbf{u}^*)$ is in the closure of the graph of the vector payoff function if $\mathbf{u}^*$ is the limit of the vector of payoffs corresponding to some sequence of strategies converging to $\mathbf{x}^*$. Player i can secure a payoff of $\beta \in \mathcal{R}$ at $\mathbf{x} \in \mathbf{X}$ if there exists $\bar{x}_i \in X_i$, such that $Z_i\left(\bar{x}_i, \sum_{j\neq i} h_j\left(x_j'\right), \alpha_i, \delta_i\right) \geq \beta$ for all $\bar{\mathbf{x}}'_{-i} \in \mathbf{X}_{-i}$ in some open neighborhood of $\mathbf{x}_{-i}$. A game is better-reply secure if whenever $(\mathbf{x}^*, \mathbf{u}^*)$ is in the closure of the graph of its vector payoff function and $\mathbf{x}^*$ is not an equilibrium, some player i can secure a payoff strictly above u_i^* at $\mathbf{x}^*$. By Theorem 3.1 in Reny (1999), if the game is compact, quasiconcave, and better-reply secure, then it possesses a pure strategy Nash equilibrium.

Proof of Proposition 3. The strategic relationship considered in the present paper satisfies all of the conditions of Reny's (1999) Theorem 3.1. First, the perturbed game is compact because even if there is no upper limit on a player's action one can focus on an appropriately chosen compact subset of the real line. Second, the payoff functions of the players are bounded. Third, consider the concavity of the players' payoff functions. If at least one of player i's opponents chooses a strictly positive action player i's payoff function is given by

$$Z_i\left(x_i, \sum_{j\neq i} h_j\left(x_j\right), \alpha_i, \delta_i\right) = \left[\delta_i\left((1-\alpha_i)\, p_i\left(x_i; \underline{\mathbf{x}}_{-i}\right) + \alpha_i p_i\left(x_i; \bar{\mathbf{x}}_{-i}\right)\right) + (1-\delta_i)\, p_i\left(x_i; \mathbf{x}_{-i}\right)\right] V_i - x_i,$$

which is continuous and concave in own strategy. The only discontinuity occurs when all of the opponents choose inaction; $\mathbf{x}_{-i} = \mathbf{0}$. Note also that this case can materialize only when $\underline{\mathbf{x}}_{-i} = 0$. Under this scenario, player i's payoff function is given by

$$Z_i\left(x_i, 0, \alpha_i, \delta_i\right) = \begin{cases} \left[\delta_i\left(1-\alpha_i\right) + \left(1-\delta_i\right)\right] \frac{V_i}{n}, & \text{if } x_i = 0 \\ \left[\delta_i\left((1-\alpha_i) + \alpha_i p_i\left(x_i; \bar{\mathbf{x}}_{-i}\right)\right) + (1-\delta_i)\right] V_i - x_i, & \text{if } x_i > 0 \end{cases},$$

which is discontinuous but concave. Thus, all players' payoff functions are concave, and hence quasiconcave, in own strategies.

It is only left to verify that the game is better-reply secure. Since the latter property is a weaker requirement than continuity, the condition for the game to be better-reply secure is satisfied at all points of continuity of a player's payoff function in own strategy, i.e. when $\sum_{j\neq i} h_j(x_j) > 0$ or when $\sum_{j\neq i} h_j(x_j) = 0$ and $x_i > 0$. Moreover, when $\mathbf{x} = \mathbf{0}$, player i's payoff function exhibits an upward jump and a strategy that slightly exceeds $x_i = 0$ can secure her a payoff that is greater than $Z_i(0, 0, \alpha_i, \delta_i) = [\delta_i(1-\alpha_i) + (1-\delta_i)]\frac{V_i}{n}$. Thus, the game is also better-reply secure, which concludes the proof of the existence of a pure strategy Nash equilibrium. ■

Proof of Proposition 4.

From the implicit function theorem,

$$\frac{\partial x^*}{\partial \delta} = -\frac{\frac{\partial F(x^*,\alpha,\delta)}{\partial \delta}}{\frac{\partial F(x^*,\alpha,\delta)}{\partial x^*}}.$$

Differentiating (7) with respect to x^* and δ, respectively, and using (7) we obtain

$$\begin{aligned}
\frac{\partial F(x^*,\alpha,\delta)}{\partial x^*} &= -\beta(n-1)V\left[\delta (x^*)^{\beta-2}\left(\begin{array}{c}\frac{(1-\alpha)\underline{x}\left((1+\beta)(x^*)^{\beta}+(1-\beta)(n-1)\underline{x}\right)}{\left((x^*)^{\beta}+(n-1)\underline{x}\right)^3}\\ +\frac{\alpha\bar{x}\left((1+\beta)(x^*)^{\beta}+(1-\beta)(n-1)\bar{x}\right)}{\left((x^*)^{\beta}+(n-1)\bar{x}\right)^3}\end{array}\right)+(1-\delta)\frac{1}{n^2(x^*)^2}\right] < 0,\\
\frac{\partial F(x^*,\alpha,\delta)}{\partial \delta} &= \frac{\beta(n-1)(x^*)^{\beta-1}}{\delta}\left[\frac{1}{V\beta(n-1)(x^*)^{\beta-1}} - \frac{1}{n^2(x^*)^{\beta}}\right]V.
\end{aligned}$$

It follows from these expressions that $\frac{\partial F(x^*,\alpha,\delta)}{\partial\delta} > 0$ if and only if $x^* > \frac{\beta(n-1)}{n^2}V = x^N$.

The equivalence between (9) and condition $x^* > x^N$ follows from an evaluation of $F(\cdot,\alpha,\delta)$ at x^N :

$$F\left(x^N,\alpha,\delta\right) = \delta\left[n^{2\beta+2}(\beta(n-1)V)^{\beta}\left(\begin{array}{c}\frac{(1-\alpha)\underline{x}^{\beta}}{\left((\beta(n-1)V)^{\beta}+n^{2\beta}(n-1)\underline{x}^{\beta}\right)^2}\\ +\frac{\alpha\bar{x}^{\beta}}{\left((\beta(n-1)V)^{\beta}+n^{2\beta}(n-1)\bar{x}^{\beta}\right)^2}\end{array}\right)-1\right]. \qquad (16)$$

■

Proof of Proposition 5.

From the implicit function theorem we have

$$\frac{\partial x^*}{\partial \alpha} = -\frac{\frac{\partial F(x^*,\alpha,\delta)}{\partial \alpha}}{\frac{\partial F(x^*,\alpha,\delta)}{\partial x^*}}.$$

From the proof of the preceding proposition we have that $\frac{\partial F(x^*,\alpha,\delta)}{\partial x^*} < 0$. Differentiating (7) with respect to α, we obtain

$$\frac{\partial F(x^*,\alpha,\delta)}{\partial \alpha} = \beta\delta V(n-1)(x^*)^{\beta-1}\left(-\frac{\underline{x}^\beta}{\left((x^*)^\beta+(n-1)\underline{x}^\beta\right)^2}+\frac{\bar{x}^\beta}{\left((x^*)^\beta+(n-1)\bar{x}^\beta\right)^2}\right),$$

which yields the first part of the proposition. The equivalence between inequalities (11) and (12) is obtained by evaluating $F(\cdot,\alpha,\delta)$ at $(n-1)^{\frac{1}{\beta}}\sqrt{\underline{x}\bar{x}}$ and comparing the resulting expression to zero. ■

Proof of Lemma 7. Application of the implicit function theorem to (13) yields

$$\frac{\partial \phi_i}{\partial x_j} = \frac{(x_i-x_j)(1-\delta_i)(x_i+\underline{x})^3(x_i+\bar{x})^3}{2\left[\delta_i(x_i+x_j)^3\left((1-\alpha_i)\underline{x}(x_i+\bar{x})^3+\alpha_i\bar{x}(x_i+\underline{x})^3\right)+(1-\delta_i)x_j(x_i+\underline{x})^3(x_i+\bar{x})^3\right]},$$

which implies the lemma since all terms in the above expression except for $(x_i - x_j)$ are positive. ■

Proof of Proposition 8. Suppose to the contrary that there exists an equilibrium in which at least one of the players incurs expenditure that exceeds the Nash level. Without loss of generality assume that $x_B^* \geq \frac{V}{4}$ and $x_B^* \geq x_A^*$.

Under the assumption that $x_B^* \geq \frac{V}{4}$, we have

$$\frac{\underline{x}}{(x_B^*+\underline{x})^2} \leq \frac{\underline{x}}{\left(\frac{V}{4}+\underline{x}\right)^2} < \frac{1}{V}. \tag{17}$$

The assumption $x_B^* \geq \frac{V}{4}$ also implies

$$\frac{\bar{x}}{(x_B^*+\bar{x})^2} \leq \frac{\bar{x}}{\left(\frac{V}{4}+\bar{x}\right)^2} < \frac{1}{V}. \tag{18}$$

Finally, the assumptions $x_B^* \geq \frac{V}{4}$ and $x_B^* \geq x_A^*$ imply

$$\frac{x_A^*}{\left(x_A^* + x_B^*\right)^2} \leq \frac{\frac{V}{4}}{\left(\frac{V}{4} + x_B^*\right)^2} \leq \frac{\frac{V}{4}}{\left(\frac{V}{4} + \frac{V}{4}\right)^2} = \frac{1}{V}. \tag{19}$$

It then follows from (17), (18) and (19) that provided $\delta_B > 0$

$$\delta_B \left(1 - \alpha_B\right) \frac{\underline{x}}{\left(x_B^* + \underline{x}\right)^2} + \delta_B \alpha_B \frac{\bar{x}}{\left(x_B^* + \bar{x}\right)^2} + \left(1 - \delta_B\right) \frac{x_A^*}{\left(x_B^* + \underline{x}\right)^2} < \frac{1}{V},$$

which implies that the equilibrium condition (13) for an interior solution cannot be satisfied. Note that the above derivations also rule out the equilibrium conditions for the corner solutions. Thus, there is no equilibrium in which at least one of the player's expenditure weakly exceeds the Nash equilibrium level. ■

Proof of Proposition 9. The derivative of a contestant's payoff with respect to own effort evaluated at a point where the two contestants choose the same effort level x is given by

$$\left(\delta \left(1 - \alpha\right) \frac{\underline{x}}{\left(x + \underline{x}\right)^2} + \delta \alpha \frac{\bar{x}}{\left(x + \bar{x}\right)^2} + \frac{\left(1 - \delta\right)}{4x}\right) V - 1. \tag{20}$$

The expression in (20) evaluated at $x = \underline{x}$ is strictly positive under our assumption that $\frac{1-\delta\alpha}{4\underline{x}} + \frac{\delta\alpha\bar{x}}{(\underline{x}+\bar{x})^2} > \frac{1}{V}$. Moreover, it is is strictly negative at $x = \bar{x}$ since $\bar{x} > \frac{V}{4} > \underline{x} > 0$. Hence, by the Intermediate Value Theorem there exists x for which the expression in (20) is equal to 0. Since the expression in (20) is strictly decreasing in x there can be only one value of $x \in (\underline{x}, \bar{x})$ for which it is equal to zero. This value of x is the effort level in the unique symmetric equilibrium. Moreover, by Lemma 10 this equilibrium is unique. ■

Proof of Lemma 10. First, note that since $\phi_i \left(x_j\right)$ attains its maximum at $x_j = x^*$, $\frac{\partial \phi_i(x^*)}{\partial x_j} = 0$ and there exists $\epsilon > 0$ such that for $x_j \in \left[x^* - \epsilon, x^* + \epsilon\right]$, $\phi_i \left(x_j\right) > x_j$ if $x_j < x^*$ and $\phi_i \left(x_j\right) < x_j$ if $x_j > x^*$ (see the proof of the following proposition for an argument demonstrating existence and uniqueness of x^*).

The claim is proved by contradiction. Suppose that there exists $\tilde{x}$, $\underline{x} < \tilde{x} < x^*$, with $\phi_i \left(\tilde{x}\right) < \tilde{x}$. But then, by the Intermediate Value Theorem, there must exist $\hat{x}$, $\tilde{x} < \hat{x} < x^* - \epsilon$ such that $\phi_i \left(\hat{x}\right) = \hat{x}$. Hence, $\hat{x}$ is a symmetric equilibrium effort level. However, this contradicts uniqueness of the symmetric equilibrium (Proposition 9). Hence, no such $\tilde{x}$ can exist which, in turn, implies the claim. A similar argument demonstrates that $x_j > \phi_i(x_j)$ when $x_j > x^*$. Since $x_i > x_j$ for $x_j \in [\underline{x}, x^*]$, Lemma 7 implies that $\phi_i(x_j)$ is strictly increasing on this interval. Similarly, $\phi_i(x_j)$ is strictly decreasing for $x_j \in [x^*, \bar{x}]$. ■

Proof of Proposition 11. The marginal benefit of player i's effort is given by

$$\left(\delta_i\left[\frac{(1-\alpha_i)\underline{x}}{(x_i+\underline{x})^2}+\frac{\alpha_i\bar{x}}{(x_i+\bar{x})^2}\right]+(1-\delta_i)\frac{x_j}{(x_i+x_j)^2}\right)V.$$

It follows from this expression that the effect of change in δ_i on this marginal benefit is proportional to

$$-\left((1-\alpha_i)\left[\frac{x_j}{(x_i+x_j)^2}-\frac{\underline{x}}{(x_i+\underline{x})^2}\right]+\alpha_i\left[\frac{x_j}{(x_i+x_j)^2}-\frac{\bar{x}}{(x_i+\bar{x})^2}\right]\right)V,$$

which implies that as long as $x_j\bar{x} > x_i^2 > x_j\underline{x}$ an increase in δ_i will decrease marginal benefit of effort. Since the marginal cost of effort is constant, an increase in δ_i will result in a decrease in player i's effort when $x_j\bar{x} > x_i^2 > x_j\underline{x}$. Hence, for points around the symmetric equilibrium, the best response function will exhibit a decrease. By Lemma 7, $\phi_j(x_i)$ is increasing for $x_i \in [\underline{x}, x]$. Hence, as a result of the shift of player i's best response curve the equilibrium will move to a point where $x_k' < x_k$ for $k \in \{A, B\}$ and $x_j' > x_i'$. ■

Proof of Proposition 14. Player A's and player B's best response functions in this case are given by

$$\phi_A(x_B)=\begin{cases}\underline{x}, \text{ if } x_B \geq \frac{(1-\delta_A)V(\underline{x}+V)^2-2\underline{x}\zeta+\sqrt{\left((1-\delta_A)V(\underline{x}+V)^2-2\underline{x}\zeta\right)^2-4\zeta^2\underline{x}^2}}{2\zeta} \\ \text{unique positive solution of} \\ \delta_A\frac{V}{(x_A+V)^2}+(1-\delta_A)\frac{x_B}{(x_A+x_B)^2}=\frac{1}{V}, \text{ otherwise}\end{cases}, \tag{21}$$

$$\phi_B(x_A)=\begin{cases}\underline{x}, & \text{if } x_A \geq \frac{(V-2\underline{x})+\sqrt{V(V-4\underline{x})}}{2} \\ \sqrt{Vx_A}-x_A, & \text{otherwise}\end{cases}, \tag{22}$$

where $\zeta \equiv (\underline{x}+V)^2-\delta_A V^2$. The result follows immediately from these expressions. ■

Proof of Proposition 15. Application of the implicit function theorem to (14) yields

$$
\begin{aligned}
\frac{dx_A^*}{d\delta_A} &= -\frac{\frac{V}{\left(x_A^*+V\right)^2}-\frac{\sqrt{Vx_A^*}-x_A^*}{Vx_A^*}}{-2\delta_A\frac{V}{\left(x_A^*+V\right)^3}-\left(1-\delta_A\right)\frac{1}{2\left(x_A^*\right)^{1.5}\sqrt{V}}} \\
&= \frac{\frac{V}{\left(x_A^*+V\right)^2}-\frac{x_B^*}{Vx_A^*}}{2\delta_A\frac{V}{\left(x_A^*+V\right)^3}+\left(1-\delta_A\right)\frac{1}{2\left(x_A^*\right)^{1.5}\sqrt{V}}}<0,
\end{aligned}
$$

since $x_B^* > x_A^*$. From (14) it also follows that

$$
\frac{dx_B^*}{d\delta_A}=\left(\frac{\sqrt{V}}{2\sqrt{x_A^*}}-1\right)\frac{dx_A^*}{d\delta_A}<0.
$$

■

Proof of Proposition 16. The interior equilibrium under ambiguity (x_A^*, x_B^*) is implicitly given by

$$
\begin{aligned}
\delta_A\frac{V}{\left(x_A^*+V\right)^2}+\left(1-\delta_A\right)\frac{x_B^*}{\left(x_A^*+x_B^*\right)^2} &= \frac{1}{V}, \\
\delta_B\frac{\underline{x}}{\left(x_B^*+\underline{x}\right)^2}+\left(1-\delta_B\right)\frac{x_A^*}{\left(x_A^*+x_B^*\right)^2} &= \frac{1}{V}.
\end{aligned}
\tag{23}
$$

Totally differentiating the first equation in (23), which implicitly defines player A's best-response function, with respect to x_A and δ_A we obtain

$$
\frac{dx_A}{d\delta_A}=-\frac{\left(Vx_B-x_A^2\right)\left(V-x_B\right)\left(x_A+V\right)\left(x_A+x_B\right)}{2\left(\delta_A V\left(x_A+x_B\right)^3+\left(1-\delta_A\right)x_B\left(x_A+V\right)^3\right)}.
$$

Hence, player A's best-response function will exhibit a local (around the equilibrium) leftward shift as a result of an increase in δ_A if and only if $\left(x_A^*\right)^2 < Vx_B^*$. Part 1 of the proposition then follows from Lemma 7 and our findings regarding the relative shapes of the two players' best-response curves.

Totally differentiating the second equation in (23), which implicitly defines player B's best-response function, with respect to x_B and δ_B we obtain

$$\frac{dx_B}{d\delta_B} = -\frac{\left(x_B^2 - x_A\underline{x}\right)\left(x_A - \underline{x}\right)\left(x_A + x_B\right)\left(x_B + \underline{x}\right)}{2\left(\delta_B\underline{x}\left(x_A + x_B\right)^3 + \left(1 - \delta_B\right)x_A\left(x_B + \underline{x}\right)^3\right)}$$

Hence, player B's best-response function will exhibit a local downward shift as a result of an increase in δ_B if and only if $x_A^*\underline{x} < \left(x_B^*\right)^2$. Part 2 of the proposition then follows from Lemma 7 and our finding regarding the relative shapes of the two players' best-response curves. ■

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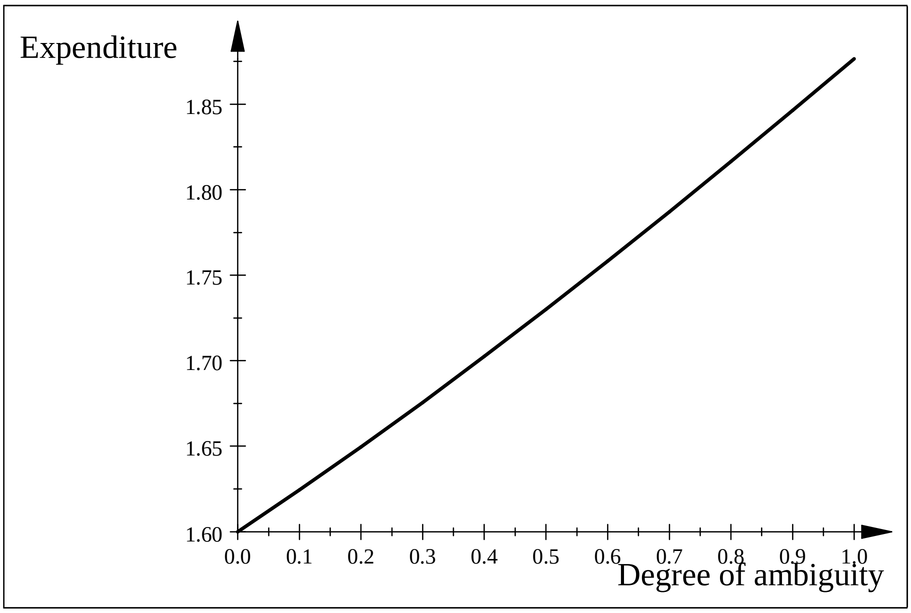

Figure 1. Equilibrium expenditure x^* as a function of the degree of ambiguity δ : ambiguity loving contenders ($n = 5$, $V = 10$, $\beta = 1$, $\underline{x} = 0.5$, $\bar{x} = 10$, $\alpha = 0.3$)

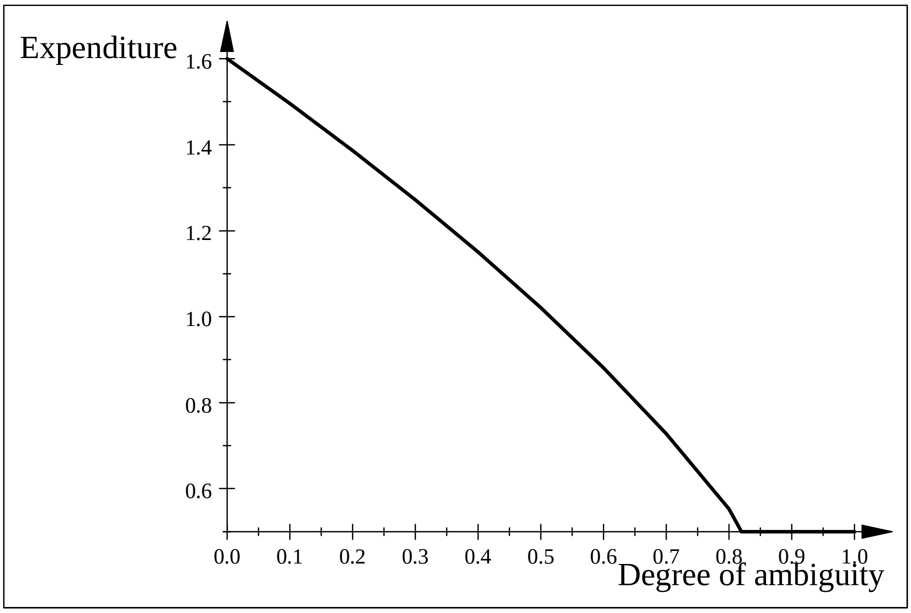

Figure 2. Equilibrium expenditure x^* as a function of the degree of ambiguity δ : ambiguity averse contenders ($n = 5$, $V = 10$, $\beta = 1$, $\underline{x} = 0.5$, $\bar{x} = 10$, $\alpha = 0.9$)

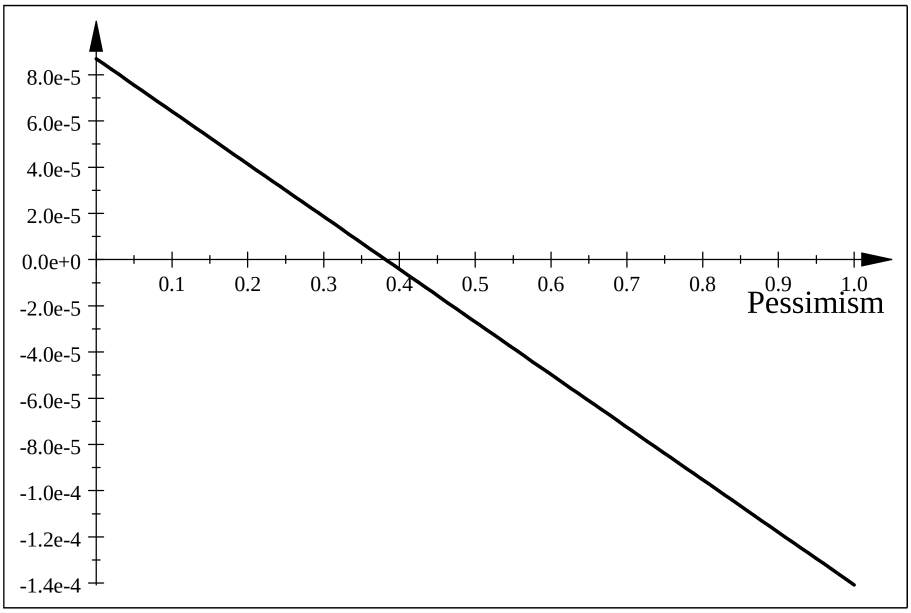

Figure 3. The equilibrium under ambiguity exceeds the Nash equilibrium for the positive values of the depicted function ($V = 10$, $\underline{x} = 0.5$, $\bar{x} = 20$, $n = 5$)

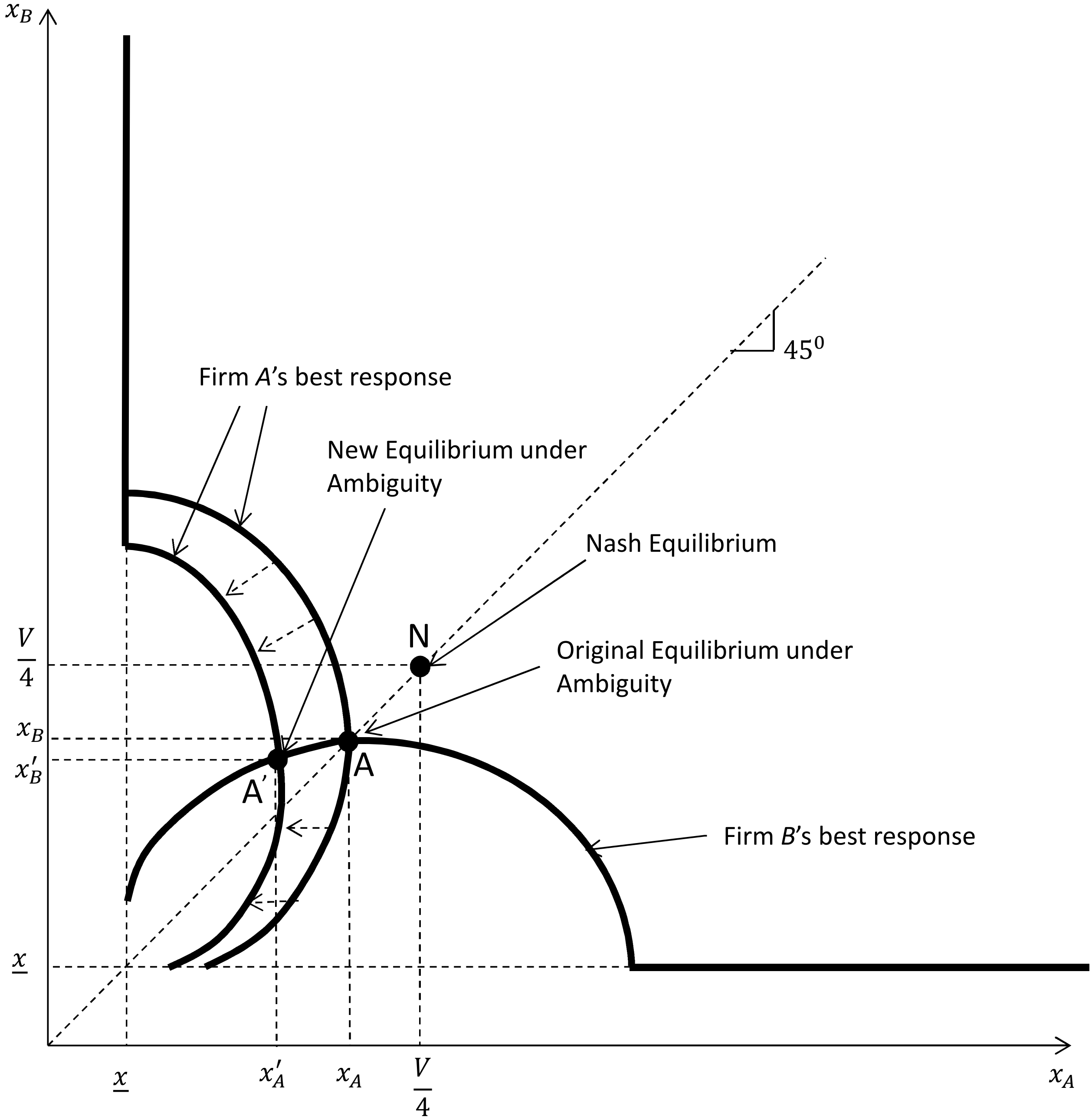

Figure 4. Player A's perception of ambiguity and equilibrium efforts

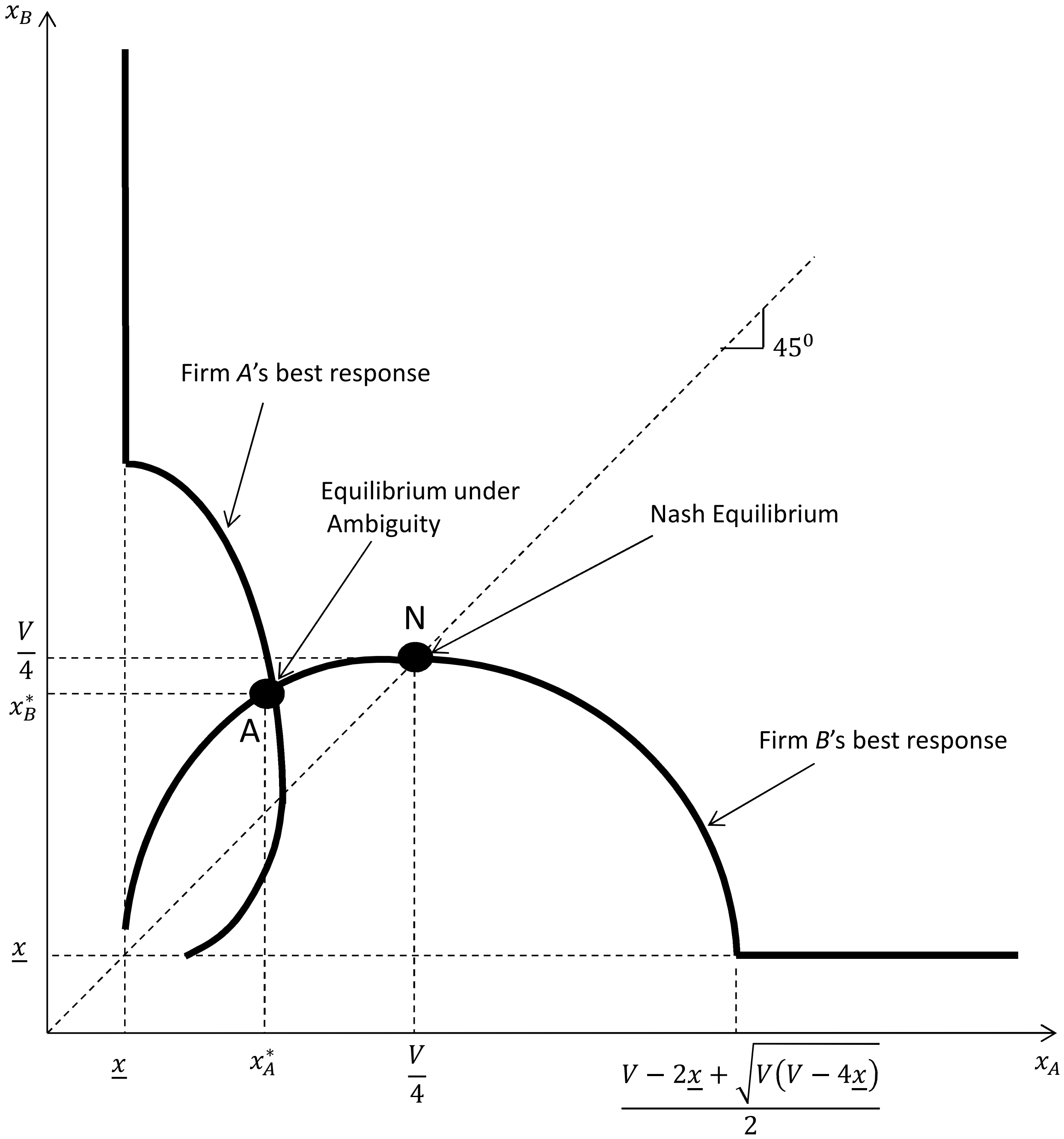

Figure 5. Player *A* is a complete pessimist and perceives small amount of ambiguity; player *B* is ambiguity neutral

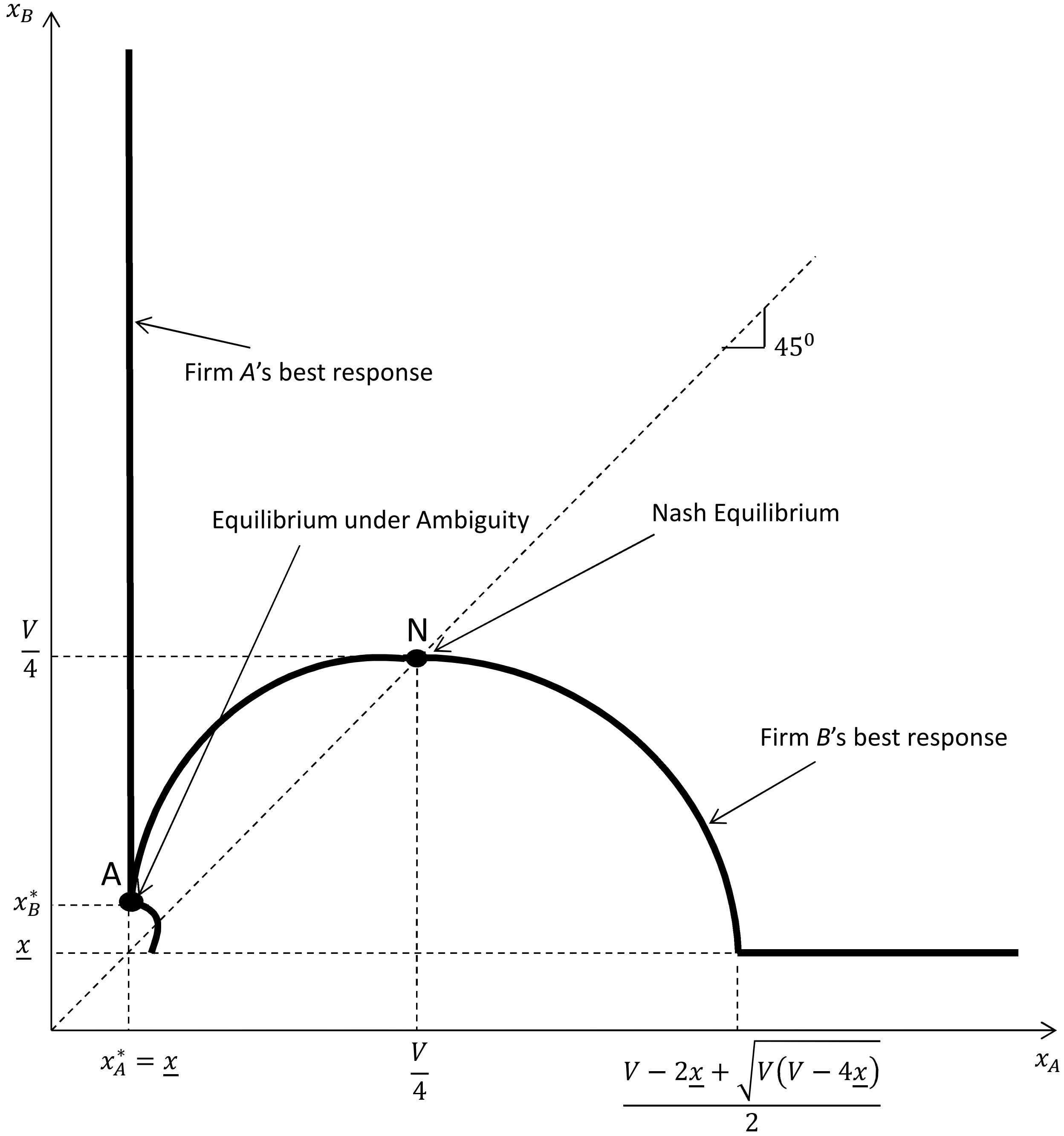

Figure 6. Player *A* is a complete pessimist and perceives a large amount of ambiguity; player *B* is ambiguity neutral

Figure 7. Equilibrium expenditures as a function of the number of ambiguity averse contenders: n = 5, V = 50, $\underline{x}$ = 3, $\delta_A = \delta_B$ = 0.3, α_A = 0.7, α_B = 0.3.