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# On the environmental Kuznets curve with fossil-fuel induced emission: Theory and some illustrative examples.

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# ABSTRACT

## On the environmental Kuznets curve with fossil-fuel induced emission: Theory and some illustrative examples.

We propose a model of fossil-fuel induced emission, which permits multiple emission-mitigation strategies. In a differentiable framework, we derive a set of necessary and sufficient conditions for an environmental Kuznets curve (EKC) in terms of the relative responses of the preference and technology-based shadow prices of emission to changes in the economic resource base when the emission policy is not allowed to adjust. Employing these conditions we construct examples of preference and technology combinations that result in an EKC in both static and dynamic frameworks. In these examples, optimal emission-mitigation strategies include employing a part of available resources for cleaning-up activities and inter-fuel substitution from dirtier to cleaner energy inputs. We show that the social optimum can be decentralised not only through standard emission policies such a Pigouvian tax, but also by a scheme that subsidises cleaning-up activities and taxes the usage of fossil fuels.

*Keywords:* Environmental Kuznets curve, marginal abatement cost, marginal willingness to pay, fossil fuels, inter-fuel substitution, abatement effort

*JEL classification codes:* Q56, Q58, H23, D62

**On the environmental Kuznets curve with fossil-fuels induced emission: Theory and some illustrative examples.**

by

Sushama Murty

## 1. Introduction.

The environmental Kuznets curve (EKC) is a central concept in the literature on growth and environment. It is seen to represent a possibility that the intuitive negative trade-off between economic growth and environmental quality can eventually be overcome as countries transition from subsistence to affluent states. The intuition behind this change is often explained in terms of changing valuation of people in favour of cleaner environment relative to material well-being as societies develop beyond subsistence levels, on the one hand, and increasing availability of more environment friendly techniques of production, on the other.

A large literature has grown around this concept. Extensive and detailed surveys of this literature include Kijima et al [2010], Dinda [2004], Stern [2004], and Dasgupta et al [2002]. A large volume of this literature is motivated by an interest to test this phenomenon empirically. Seminal contributions include works by Grossman and Krueger [1993, 1995], Shafik and Bandyopadhyay [1992], Holtz-Eakin and Seldon [1992], Panayotou [1993], and Seldon and Song [1993]. The evidence seems mixed. EKC appears to hold for pollutants with more localised impacts such as sulphur dioxide and nitrous oxide emissions. Emissions whose impacts are more diffused, such as several green house gasses, *e.g.,* carbon dioxide, have continued to rise with growth.[1]

A significant literature also aims to explain this phenomenon by employing creative theoretical models. The main aim of this literature is to explore and identify conditions under which this phenomenon becomes theoretically plausible. To provide a motivation for this paper, it will be convenient to classify the existing theoretical literature on EKC on the basis of how emission generation is modelled in these papers. Since it is difficult to do justice to all papers, we have chosen a subset to bring out the diversity that exists in the way emission generation is modelled. In the dynamic frameworks adopted in papers such as Stokey [1998] and Hartman and Kwon [2005], emission is a result of production of physical output. In addition, in Stokey, the emission so generated can be reduced when a part of the physical output is allocated towards abatement efforts, while in Hartman and Kwon, physical capital can be used for abating emission. Several papers such as John and Pecchenino [1994], McConnell [1997], Andreoni and Levinison [2001], Lieb

[1] See, *e.g.,* the World Bank Report [1992]. See also Mason and Swanson [2001] for evidence regarding chloroflurocarbons.

[2002], Plassman and Khanna [2006], and Egli and Steger [2007] treat emission as a consumption externality, *i.e.,* emission is generated as result of consumption activities in these papers. At the same time, a part of the economy's endowment can be allocated for abatement purposes. Seldon and Song [1995] employs Forster's [1973] model where emission generation is a result of the use of physical capital. Jones and Manuelli [2001] assume pollution is generated by a continuum of capital types differentiated on the basis of their costs and the amount of pollution that they generate, although capital types are assumed to be perfectly substitutable in production. Lopez [1994] and Lopez and Mitra [2000] do not model emission generation explicitly. Rather, they consider the environmental factor as another productive input into the production process. The terms pollution and environmental factor are used interchangeably to signify the fact that the demand for the environmental factor in production is a demand for nature's services for disposing off the pollution generated by the production process.

In the context of current debates on climate change, the most significant source of anthropogenic emissions is the combustion of fossil fuels. Considerable real-world data exists on the magnitudes of the use of these energy inputs and data on gross emission is often generated based on the amounts of fossil-fuels combusted in the production process.[2] However, to the best of our knowledge, fossil fuels or other direct sources of anthropogenic emission are not commonly included in the existing theoretical works that study the EKC. In this paper, we aim to bridge this gap by proposing a model where emission is generated due to the combustion of fossil fuels, which are inputs into production of the economy's output, in addition to standard inputs such as capital. The model allows for a range of energy inputs that differ with respect to costs of extraction and their emission generating intensities. Hence, such a model permits a range of emission-mitigation strategies. In addition to allocating a part of the economy's output to cleaning-up effort (as is standard in the literature mentioned above), emission can also be mitigated by reducing usage of fossil fuels and substituting dirtier by cleaner energy inputs. It is clear that the design of optimal emission policy implies also the design of the optimal mix of strategies available to mitigate emission. This leads one to conjecture if a Pigouvian tax can be substituted by policies that aim to control emission by controlling its sources or by encouraging its mitigation. Indeed, we find that the social optimum can be decentralised not only through standard emission policies such a Pigouvian tax, but also by a scheme that subsidises abatement activities and taxes the usage of fossil fuels.

In the context of such a model, we study the plausibility of an EKC arising in both the static

[2] See IPCC's Tier 1 Sectoral Approach which is used by data reported *e.g.,* by the International Energy Agency.

and dynamic contexts. To do so, we first derive a set of necessary and sufficient conditions that characterise an EKC in a basic framework, to which most models in the literature, including our proposed model of fossil-fuel induced emission, can be reduced.[3] These conditions are expressed in terms of the relative responses of preference and technology-based shadow prices of emission in a situation when the productive capacity of an economy changes but the emission policy does not adjust. Lieb [2002] also derives a set of similar types of conditions in the static case when emission is a consumption externality. Our analyses in this and in a companion paper Murty [2014] show that these conditions hold more generally.[4] We believe that the scope offered by these conditions has not been explored sufficiently in Lieb [2002], and that they provide valuable insights for the study and identification of various preference and technology combinations that can result in the EKC phenomenon. In this paper, we provide some examples of such preference and technological structures in a model where emission is caused by the use of fossil fuels.

In this paper, as in many papers in the theoretical literature, an EKC is a possible outcome of the comparative statics of a welfare maximisation problem subject to a feasibility constraint, where the locus of optimal emission level is traced when the productive capacity of an economy (as measured by the level of its economic resource) changes. In Section 2, we describe the basic framework that we will employ to derive the necessary and sufficient conditions for the EKC in a differentiable world. Section 3 proposes a model with fossil-fuel induced emission and shows how it can be reduced to the basic framework in Section 2. Section 4 derives the necessary conditions of welfare maximisation. Section 5 derives the necessary and sufficient conditions for the EKC phenomenon.[5] These conditions are employed in Sections 6 and 7 to identify some preference and technology structures that result in an EKC in the case when emission is caused by the use of fossil fuels. In particular, Section 6 considers a static case, while Section 7 considers a dynamic growth model. In Section 8, we discuss decentralisation of the social optimum through suitable taxation and subsidisation schemes. We conclude in Section 9.

## 2. A basic framework of analysis in the emission-consumption space.

In this section, we present a general framework of analysis, which we will employ to study the necessary and sufficient conditions for an EKC. Many of the existing models in the literature, which differ with regards to how emission is generated, either adopt this framework or (we

[3] We demonstrate this for some standard models in a companion paper Murty [2014].

[4] In this paper, all functions are differentiable. In Murty [2014], these conditions are derived in a more general framework that allows non-smooth economies.

[5] These condition are derived also in Murty [2014] as a special case of a more general framework that permits both smooth and non-smooth economies. Here, we offer a simpler and a more direct proof in the case of smooth economies.

conjecture) can be reduced to it. We have shown this to be true for some models in a companion paper Murty [2014]. We will show in the following sections how a model with fossil-fuel induced emission can also be reduced to this basic framework. The two main components of this framework are preferences defined over emission and consumption and the set of feasible emission and consumption combinations made possible by the constraints that the economy faces.[6]

### *2.1. Preferences.*

Preferences are defined over two goods, emission and consumption. The quantities of emission and consumption are denoted by $z \in \mathbf{R}_+$ and $c \in \mathbf{R}_+$, respectively.

**Assumption 1.** *Preferences are represented by a utility function $u : \mathbf{R}^2_+ \longrightarrow \mathbf{R}_+$, which is strictly quasi-concave and twice continuously differentiable with $u_z < 0$ and $u_c > 0$.*[7]

The preference marginal rate of substitution (MRS) between emission and consumption (or the preference-based shadow price of emission) is defined by the function $r : \mathbf{R}^2_{++} \longrightarrow \mathbf{R}_{++}$ with image

$$r(z,c) = -\frac{u_z(z,c)}{u_c(z,c)}. \tag{2.1}$$

This is the marginal willingness to pay (MWTP) for emission reduction.[8] The determinant of the bordered Hessian of $u$ is defined as the function $Q^u : \mathbf{R}^2_{++} \longrightarrow \mathbf{R}$ with image $Q^u(z,c)$. Strict quasi-concavity of function $u$ implies that $Q^u$ takes strictly positive values. The following relation between functions $Q^u$ and $r$, which we will employ in the proof of Theorem 2, below, can be verified in a straightforward manner:

$$\frac{Q^u(z,c)}{u_c^2} = -u_z \frac{\partial r(z,c)}{\partial c} + u_c \frac{\partial r(z,c)}{\partial z}. \tag{2.2}$$

### *2.2. The feasible set of emission and consumption.*

The feasible set of emission and consumption combinations depends on a parameter $y$, which is very broadly defined. In this paper, we will assume that the economic-resource base is defined by the level of the capital stock. The analysis will be conducted in both static and dynamic cases. In the static case, we will interpret $y$ as the capital stock. In the dynamic case, the capital stock

[6] In general, these constraints could be due to scarcity, technological, institutional, etc. In this paper, we focus on the technological constraints imposed by standard production relations that transform inputs into intended outputs and the constraints imposed by nature in transforming some emission-causing inputs such as fossil fuels into emission.

[7] Given $S \subseteq \mathbf{R}^n$, a function $f : S \longrightarrow \mathbf{R}$ is differentiable if it is differentiable in the interior of its domain $S$.

[8] Alternatively, it is the marginal willingness to accept compensation for increase in emission.

and, hence, the set of feasible emission-consumption combinations changes over time. Hence, in this case, we will interpret $y$ as time.

Given $y \geq 0$, the feasible set of emission-consumption combinations is denoted by $Y(y)$. See Figure 1 for a diagrammatic example. We will also define the set of feasible consumption levels for given levels of both $y$ and emission as the set

$$Y(z, y) := \{c \geq 0 \mid \langle z, c\rangle \in Y(y)\} \quad \forall \quad \langle z, y\rangle \in \mathbf{R}^2_+. \tag{2.3}$$

The assumption below specifies a functional representation of the feasible set of emission-consumption combinations that, as we will show in Section 3, is relevant for a model where emission is a production externality caused by the use of fossil fuels.

**Assumption 2.** *There exists function $F : \mathbf{R}^3_+ \longrightarrow \mathbf{R}$ such that, for every $y \geq 0$, $Y(y)$ can be represented functionally as the set*

$$Y(y) = \left\{\langle z, c\rangle \in \mathbf{R}^2_+ \mid F(z, c, y) \leq 0\right\},$$

*where $F$ is twice continuously differentiable and quasi-convex with $F_z < 0$ and $F_c > 0$. If $z > 0$ and $y > 0$ then $Y(z, y) \cap \mathbf{R}_{++} \neq \emptyset$. $\langle 0, 0\rangle \in Y(y)$ for all $y \geq 0$.*

Given the set of all feasible emission-consumption combinations for $y \geq 0$, efficiency requires the maximal production of the consumption good and the minimal production of emission. Hence, we define the strictly efficient frontier of $Y(y)$ for $y \geq 0$ as the set

$$\left\{\langle z, c\rangle \in Y(y) \;\middle|\; \nexists \langle z', c'\rangle \in Y(y) \text{ such that } \langle z', c'\rangle \neq \langle z, c\rangle,\ z' \leq z,\ \text{and } c' \geq c\right\}.$$

Thus, given $y \geq 0$, $\langle z, c\rangle \in Y(y)$ is a *strictly efficient point of* $Y(y)$ if there is no other emission-consumption combination in $Y(y)$ with consumption level greater than or equal to $c$ and emission level less than or equal to $z$. For all $y \geq 0$, under Assumption 2, it can be verified that $\langle z, c\rangle \in \mathbf{R}^2_+$ is a strictly efficient point of $Y(y)$ if $F(z, c, y) = 0$.

Define function $s : \mathbf{R}^3_{++} \longrightarrow \mathbf{R}_{++}$ with image

$$s(z, c, y) = -\frac{F_z(z, c, y)}{F_c(z, c, y)}. \tag{2.4}$$

If $\langle z, c, y\rangle$ belongs to the strictly efficient frontier of $Y(y)$, then $s(z, c, y)$ defines the technological MRS between emission into consumption at $\langle z, c, y\rangle$. Under Assumption 2, this trade-off between emission and consumption $s(z, c, y)$ is positive. It denotes the marginal abatement cost (MAC) of emission reduction.[9] For every $y > 0$, the determinant of the bordered Hessian of function

[9] Alternatively, it denotes the marginal return, in terms of consumption, from an increase in emission.

$F$ with respect to consumption and emission is defined as the function $Q^F : \mathbf{R}^3_{++} \longrightarrow \mathbf{R}$ with image $Q^F(z, c, y)$.[10] Quasi-convexity of function $F$ in $c$ and $z$ implies that $Y(y)$ is a convex set for all $y \geq 0$ and that $Q^F(z, c, y) \leq 0$. It can be also be verified that the following relation, which we will employ later in the proof of Theorem 2, holds:

$$\frac{Q^F(z, c, y)}{F_c^2} = -F_z \frac{\partial s(z, c, y)}{\partial c} + F_c \frac{\partial s(z, c, y)}{\partial z}. \tag{2.5}$$

## 3. Emission as a production externality generated by fossil fuels.

In this section, we propose a model where emission is generated as a result of the use of fossil-fuels. From this model, we will derive the feasible set of emission and consumption for any given level of $y$ in both static and dynamic contexts.

### *3.1. A model with fossil-fuel induced emission.*

The approach is similar to Murty et al [2012], where emission is a by-product of the production of the intended output. The production of the intended output requires the use of certain emission-causing inputs such as fossil fuels. At the same time, in nature, emission is a result of the combustion of fossil fuels. Thus, the use of fossil-fuel in the production of the intended output results also in emission generation. The production relation that characterises intended-output production is distinct from nature's emission-generating technology.

We assume that there are $n$ emission-causing fossil fuels indexed by $j$. The amounts of fossil fuels used during production will be denoted by a vector $x \in \mathbf{R}^n_+$. The per-unit extraction cost (or price) of the $j^{th}$ fossil-fuel is $w_j \geq 0$ and $w \in \mathbf{R}^n_+$ is the vector of per unit costs of the fossil fuels. Capital, denoted by $k$, and fossil fuels are used as inputs to produce an intended output. Inputs are converted into the intended output by a production function $f : \mathbf{R}^2_+ \longrightarrow \mathbf{R}_+$ with image $f(k, x)$. The output so produced can be used for consumption $c \geq 0$, abatement $a \geq 0$, investment $i \geq 0$, and for meeting the costs of acquiring fossil fuels $wx \geq 0$. Thus, with free disposability of the intended output, we have

$$f(k, x) \geq c + a + i + wx.$$

[10] Thus, $Q^F(z, c, y)$ is the determinant of the $3 \times 3$ matrix

$$\begin{bmatrix} 0 & F_z(z,c,y) & F_c(z,c,y) \\ F_z(z,c,y) & F_{zz}(z,c,y) & F_{zc}(z,c,y) \\ F_c(z,c,y) & F_{zc}(z,c,y) & F_{cc}(z,c,y) \end{bmatrix}.$$

**Assumption 3.** *$f$ is a twice continuously differentiable, increasing, and strictly concave function.*[11]

The use of fossil fuels in intended production leads to *gross* emission generation in nature. In our model there are three ways of mitigating emissions, namely, by reducing the usage of fossil-fuels in intended production, by substituting usage of dirtier by cleaner fuel inputs, and by undertaking direct abatement measures. Nature's emission generating technology relates the usage of fossil fuels and the abatement effort to the amount of *net* emission produced. Define $E \subset \mathbf{R}_+^{n+2}$ as the set of all emission, fossil-fuels, and abatement levels that are feasible under nature's emission generating mechanism.

We say that $\langle z, x, a\rangle \in E$ is a strictly efficient point of $E$ if there does not exist $\langle z', x', a'\rangle \in E$ such that $\langle z', x', a'\rangle \neq \langle z, x, a\rangle$ with $z' \leq z$, $x' \geq x$, and $a' \leq a$.[12] It is intuitive that a welfare maximising planner (who will act to minimise emission generation and maximise intended output production) will choose an emission, fossil-fuels, and abatement combination that will lie on the strictly efficient frontier of nature's technology, $E$. Assumption 4 presents a functional representation of set $E$.

**Assumption 4:** *There exists a function $G : \mathbf{R}_+^{n+2} \longrightarrow \mathbf{R}$ with image $G(z, x, a)$ such that $G$ is strictly quasi-convex and twice continuously differentiable with $G(0, 0^n, 0) = 0$, $G_a < 0$, $G_z < 0$, $G_{x_j} > 0$ for all $j = 1, \ldots, n$, and*

$$E = \left\{ \langle z, x, a\rangle \in \mathbf{R}_+^{n+2} \middle| \; G(z, x, a) \leq 0 \right\}.$$

Note that $\langle z, x, a\rangle \in E$ is a strictly efficient point of $E$ if and only if $G(z, x, a) = 0$ and that, along the strictly efficient frontier of $E$, the trade-offs between emission and abatement, emission and fossil fuels, and abatement and fossil-fuel usage are given, respectively, by $-\frac{G_z}{G_a} < 0$, $-\frac{G_z}{G_{x_j}} > 0$, and $-\frac{G_a}{G_{x_j}} > 0$ for all $j = 1, \ldots, n$, and that these signs satisfy our intuition about the relationship between these goods in terms of nature's emission generating mechanism.[13] Further, Assumption

[11] Note, that function $f$ can take several forms capturing different types of relationship between capital and fossil fuels as inputs in production. It permits any degree of substitutability or complementarity between them.

[12] This implies that, intuitively, $\langle z, x, a\rangle \in E$ is a technically *inefficient* point of nature's emission generating technology if nature's technology permits the same amount of emission to be produced with smaller amounts of abatement effort or bigger amounts of fossil fuels or if the same amounts of abatement effort and fossil fuels can produce a smaller amount of emission. See Murty et al [2012] for the disposability properties of set $E$.

[13] If net emission level is to be held fixed, then $-\frac{G_a}{G_{x_j}} > 0$ implies that an increase in the usage of the $j^{th}$ fossil-fuel needs to be accompanied by an increase in efforts to abate emissions. $-\frac{G_z}{G_{x_j}} > 0$ implies that, in nature, combustion of more fossil fuels generates more emission if abatement effort is held fixed. $-\frac{G_z}{G_a} < 0$ implies that more abatement effort reduces emission when fossil-fuel usage is held fixed.

4 implies that the efficient amount of emission is zero if fossil fuels are not employed and if no abatement is undertaken. Examples of function $G$ can be seen in Sections 3.2.2, 6, and 7.

Often, it will be possible and convenient to assume that function $G$ is globally invertible in emission or abatement, *i.e.,* there exists a function $\Phi : \mathbf{R}_+^{n+1} \longrightarrow \mathbf{R}_+$ with image $a = \Phi(z, x)$ such that $G(z, x, \Phi(z, x)) = 0$ or there exists a function $\Theta : \mathbf{R}_+^{n+1} \longrightarrow \mathbf{R}_+$ with image $z = \Theta(x, a)$ such that $G(\Theta(x, a), x, a) = 0$. In other words,

$$G(z, x, a) = 0 \iff a = \Phi(z, x) \iff z = \Theta(x, a). \tag{3.1}$$

Intuitively, nature's emission-generating technology specifies the strictly efficient level of abatement that is consistent with $z$ amount of net emission when $x$ amount of fossil fuels are burnt or specifies the strictly efficient amount of net emission resulting from combusting $x$ amount of fossil fuels and exerting $a$ amount of abatement effort to clean up emission.

*3.2. Deriving the set of feasible emission-consumption combinations in the static case and an example.*

In this section, we derive the set of feasible emission-consumption combinations in a static case with no capital accumulation and give an example that satisfies Assumption 2.

*3.2.1. Deriving the set of feasible set of emission-consumption combinations in the static case.*

In the static case, investment $i$ is zero and we interpret $y$ as the economic resource-base, whose level is given by the amount of the capital input. The entire output produced by the use of inputs is allocated to meet fossil-fuel costs and for consumption and abatement activities. Given the level of capital $k$, for every level of emission $z$, the set of feasible consumption levels can be derived by first computing function $c^m : \mathbf{R}_+^2 \longrightarrow \mathbf{R}_+$, which is defined as

$$\begin{aligned} c^m(z, k) &:= \max_{\langle c, x, a\rangle \in \mathbf{R}_+^3} \left\{ c \geq 0 \;\middle|\; f(k, x) \geq c + a + wx,\ \langle z, x, a\rangle \in E \right\} \\ &\iff \max_{\langle x, a\rangle \in \mathbf{R}_+^2} \left\{ f(k, x) - a - wx \;\middle|\; G(z, x, a) \leq 0 \right\}. \end{aligned} \tag{3.2}$$

Thus, given a stock of capital, function $c^m$ defines the maximum consumption that is feasible for every level of emission. Hence, for every $k \geq 0$ and $z \geq 0$, since the intended output is freely disposable, all the feasible consumption levels lie in the interval $[0, c^m(z, k)]$. Hence, given $k \geq 0$, the set of feasible emission-consumption combinations is defined by

$$Y(k) := \left\{ \langle z, c\rangle \in \mathbf{R}_+^2 \;\middle|\; c \leq c^m(z, k) \right\}. \tag{3.3}$$

This implies that the image of function $F$ in Assumption 2 is defined by

$$F(z, c, k) := c - c^m(z, k) \;\text{ so that }\; F(z, c, k) \leq 0 \iff c \leq c^m(z, k). \tag{3.4}$$

In particular, if $c^m$ is quasi-concave, then $F$ is quasi-convex.

#### *3.2.2. A simple example.*

To simplify computations and to arrive at a closed-form functional representation of the feasible set of emission-consumption combinations, in this example, we assume that function $f$ is additively separable.[14] Suppose there is only one fossil fuel ($n = 1$) and the technology is specified by

$$f(k, x) = k^{\frac{1}{2}} + x \quad \text{and} \quad G(z, x, a) = \frac{x^2}{a+2} - z.$$

Note, $f$ and $G$ satisfy Assumptions 3 and 4.[15] Solving (3.2) for this case (assuming $w < 1$) yields

$$c^m(z, k) = \frac{1}{4}\left(4k^{\frac{1}{2}} + 8 + (w-1)^2 z\right).$$

Note that $c^m$ is quasi-concave. The set of feasible emission-consumption combinations can be derived from (3.3) and function $F$ is derived from (3.4). Note that $F$ satisfies all properties in Assumption 2.

### *3.3. Deriving the set of feasible emission-consumption combinations in the dynamic case.*

In the dynamic case, capital stock changes over time. Let time be denoted by $t \geq 0$. A given trajectory $k(t)$ of the capital stock defines the trajectory of investment $i(t) = \dot{k}(t) - \delta k(t)$, where $\delta \geq 0$ is the rate of depreciation of the capital stock. Below, we show that the set of feasible emission-consumption combinations varies with time as the stock of capital and investment vary with time. Hence, in the dynamic case, we interpret $y$ as time so that, given a trajectory of capital $k(t)$, $Y(t)$ defines the set of all emission-consumption combinations that are made feasible by the level of the capital stock $k(t)$ at time $t$.

Given a trajectory of capital $k(t)$, define function $c^m : \mathbf{R}^2_+ \longrightarrow \mathbf{R}_+$ as the outcome of the following static problem at time $t$ and recalling (3.1):

$$\begin{aligned} c^m(z,t) &:= \max_{\langle c,x,a\rangle \in \mathbf{R}^{n+2}_+} \left\{ c \geq 0 \;\middle|\; f(k(t),x) \geq \delta k(t) + \dot{k}(t) + c + a + wx,\ G(z,x,a) \leq 0 \right\} \\ &\iff \max_{x \in \mathbf{R}^n_+} \left\{ f(k(t),x) - \delta k(t) - \dot{k}(t) - \Phi(z,x) - wx \right\}. \end{aligned} \tag{3.5}$$

Thus, $c^m(z, t)$ gives the maximum level of consumption that is possible at time $t$, when the stock of capital is $k(t)$ and the emission level is fixed at $z$. Let $\tilde{x}(z, t)$ be the solution of Problem (3.5).

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[14] Later, we also provide an example where $f$ has a Cobb Douglas form and where we obtain an EKC.

[15] Note that, in this case, $G(z, x, a) = 0 \iff z = \Theta(x, a) \equiv \frac{x^2}{a+2}$, *i.e.*, in nature, emission increases as fossil fuel usage increases and abatement decreases. In particular, the case when no abatement is done, *i.e.*, $a = 0$, corresponds to the maximal level of (gross) emission ($\frac{x^2}{2}$) for any given level $x$ of fossil fuel.

Hence, $c^m(z,t) = f(k(t), \tilde{x}(z,t)) - \delta k(t) - \dot{k}(t) - \Phi(z, \tilde{x}(z,t)) - w\tilde{x}(z,t)$. As in the static case, for every $t \geq 0$, the set of feasible emission-consumption combinations is defined by

$$Y(t) := \left\{ \langle z, c \rangle \in \mathbf{R}^2_+ \;\middle|\; c \leq c^m(z,t) \right\}. \tag{3.6}$$

The image of function $F$ in Assumption 2 is

$$F(z,c,t) := c - c^m(z,t) \;\text{ so that }\; F(z,c,t) \leq 0 \iff c \leq c^m(z,t). \tag{3.7}$$

## 4. The welfare maximisation problem and its necessary conditions.

The planner maximises welfare subject to a feasibility constraint. The solution function of our concerned welfare maximisation problem is $\varphi : \mathbf{R}_+ \longrightarrow \mathbf{R}^2_+$ with image defined as

$$\varphi(y) \equiv \langle \hat{z}(y), \hat{c}(y) \rangle := \arg \max_{\langle z,c \rangle \in \mathbf{R}^2_+} \{u(z,c) \big| \; \langle z, c \rangle \in Y(y)\}. \tag{4.1}$$

The Lagrangian of the welfare maximsation problem (4.1) is

$$L = u(z,c) - \lambda F(z,c,y),$$

where $\lambda$ is the relevant Lagrange multiplier. We will assume that the problem generates interior solutions, *i.e.,* $\hat{z}(y) > 0$ and $\hat{c}(y) > 0$ for $y > 0$.[16] The Kuhn-Tucker first-order necessary conditions (which are also sufficient for this convex optimisation problem) yield, for $y > 0$,

$$\begin{aligned} &(i) \quad u_z(z,c) - \lambda F_z(z,c,y) = 0, \\ &(ii) \quad u_c(z,c) - \lambda F_c(z,c,y) = 0 \text{ and} \\ &(iii) \quad F(z,c,y) \leq 0, \; \lambda F(z,c,y) = 0, \; \lambda \geq 0. \end{aligned} \tag{4.2}$$

We can verify that a solution to (4.1) is strictly efficient (*i.e.,* $F(z,c,y) = 0$) and $\lambda > 0$.[17] Since we have assumed that Problem (4.1) has only interior solutions, (i) and (ii) of (4.2) imply equality of the preference and technological MRS between emission and consumption:

**Theorem 1:** *For every $y > 0$, if $\varphi(y) = \langle z, c \rangle$ then*

$$r(\boldsymbol{z}, \boldsymbol{c}) \equiv -\frac{u_z(\boldsymbol{z}, \boldsymbol{c})}{u_c(\boldsymbol{z}, \boldsymbol{c})} = -\frac{F_z(\boldsymbol{z}, \boldsymbol{c}, y)}{F_c(\boldsymbol{z}, \boldsymbol{c}, y)} \equiv s(\boldsymbol{z}, \boldsymbol{c}, y),$$

*where $\boldsymbol{z} = \hat{z}(y)$ and $\boldsymbol{c} = \hat{c}(y)$.*

16 This will be true for our (non-pathological) examples in Sections 6 and 7.

17 For, if $F(z,c,y) < 0$, then (iii) implies $\lambda = 0$. Hence, (i) implies $u_z(z,c) = 0$, which contradicts the assumption $u_z(z,c) < 0$. If $\lambda = 0$, then (i) once again implies $u_z(z,c) = 0$, which contradicts the assumption $u_z(z,c) < 0$.

## 5. Necessary and sufficient conditions for an environmental Kuznets curve (EKC).

In this section, we define an environmental Kuznets curve (EKC) and provide a set of necessary and sufficient conditions for this phenomenon to happen. An EKC is a possible outcome of the comparative statics of the welfare maximisation problem (4.1), where the locus of the optimal emission level is traced when the feasible set of emission and consumption changes because of changes in the variable $y$.

**Definition:** The function $\varphi$ exhibits an EKC if there exists $\overset{*}{y} > 0$ such that

(i) $\frac{\partial \hat{z}(y)}{\partial y} > 0$ whenever $0 < y < \overset{*}{y}$,

(ii) $\frac{\partial \hat{z}(y)}{\partial y} < 0$ whenever $y > \overset{*}{y}$ and

(iii) $\frac{\partial \hat{z}(\overset{*}{y})}{\partial y} = 0$.

For convenience in exposition but without loss of generality (WOLOG), in this section, we will interpret $y$ as the level of the economic resource base.[18] The effect of a change in the resource base $y$ on the optimal level of emission can be decomposed into: (i) an *emission-held-fixed effect (EHFE)*, where the emission policy does not adjust and only consumption adjusts to the change in $y$ and (ii) a *difference-in-the shadow-prices effect (DISPE)*, where the emission policy also adjusts optimally depending on the relationship between the MAC and the MWTP at the outcome reached by the EHFE. The necessary and sufficient conditions for an EKC pertain to the relative changes in the MAC and the MWTP when we move from the initial optimum to the outcome reached by the EHFE.

To state these necessary and sufficient conditions for an EKC, it will be helpful to define function $c^m : \mathbf{R}^2_+ \longrightarrow \mathbf{R}_+$, which assigns the maximum level of consumption that is possible to every $y \geq 0$ and $z \geq 0$:[19]

$$c^m(z, y) := \max \{c \geq 0 \mid c \in Y(z, y)\}.$$

We now describe some pertinent features of function $c^m$. Note, Assumption 2 implies that $c^m(z, y) > 0$. It also implies that $\langle z, c^m(z, y)\rangle$ is a strictly efficient point of $Y(y)$. Hence,

$$F(z, c^m(z, y), y) = 0 \iff c = c^m(z, y)$$

and, for $y > 0$, we obtain the following from the implicit function theorem:

$$\frac{\partial c^m(z, y)}{\partial z} = -\frac{F_z(z, c, y)}{F_c(z, c, y)} \text{ and } \frac{\partial c^m(z, y)}{\partial y} = -\frac{F_y(z, c, y)}{F_c(z, c, y)}, \text{ where } c = c^m(z, y). \tag{5.1}$$

[18] Similar arguments will follow if $y$ is interpreted as time in a dynamic context.

[19] To recall the definition of $Y(z, y)$, see (2.3).

Given that function $u$ is increasing in $c$, it is intuitive that $c^m(z,y)$ is also the solution of a restricted welfare maximisation problem, where both the resource and emission levels are held fixed.

$$c^m(z,y) = \arg\max_{c\geq 0} \{u(z,c) \mid c \in Y(z,y)\}. \tag{5.2}$$

It is clear that Assumption 1 and Problems (4.1) and (5.2) imply that, for $y \geq 0$, if emission is held fixed at the optimal level $\hat{z}(y)$, then $c^m(\hat{z}(y), y)$ is equal to the optimal level of consumption corresponding to $y$, *i.e.,*

$$\hat{c}(y) = c^m(\hat{z}(y), y). \tag{5.3}$$

We now employ function $c^m$ to explain the necessary and sufficient conditions for an EKC in terms of the EHFE and the DISPE. Starting from an initial optimum $\varphi(y) = \langle \hat{z}(y), \hat{c}(y)\rangle =: \langle \boldsymbol{z}, \boldsymbol{c}\rangle =: A$ corresponding to an initial level of resource base $y > 0$, the EHFE of a change in the level of the resource base is given by $\frac{\partial c^m(\boldsymbol{z},y)}{\partial y}$. It reflects the maximum change in consumption due to a change in the level of the resource base when the level of emission is held fixed at the initial optimal level $\boldsymbol{z}$, *i.e.,* when the emission policy does not adjust to the change in the resource base and only consumption adjusts to solve the Problem (5.2). In Figure 1, the EHFE due to a change in the resource base from $y$ to $y'$ is reflected by the movement from point $A$ to point $B$.

Note that (2.4) and (5.1) imply

$$s(z, c^m(z,y), y) = \frac{\partial c^m(z,y)}{\partial z}. \tag{5.4}$$

Employing (5.4), define functions $dS : \mathbf{R}^2_{++} \longrightarrow \mathbf{R}_+$ and $dR : \mathbf{R}^2_{++} \longrightarrow \mathbf{R}_+$ as

$$\begin{aligned} dS(z,y) &:= \frac{\partial s(z, c^m(z,y), y)}{\partial c}\frac{\partial c^m(z,y)}{\partial y} + \frac{\partial s(z, c^m(z,y), y)}{\partial y} = \frac{\partial^2 c^m(z,y)}{\partial y \partial z} \\ dR(z,y) &:= \frac{\partial r(z, c^m(z,y))}{\partial c}\frac{\partial c^m(z,y)}{\partial y}. \end{aligned} \tag{5.5}$$

Suppose $\langle z, c\rangle$ is a strictly efficient point of $Y(y)$, *i.e.,* $F(z,c,y) = 0$ or, equivalently, $c = c^m(z,y)$. Then, starting from this point, $dS(z,y)$ and $dR(z,y)$ define, respectively, the changes in the MAC and the MWTP when the resource base changes and only consumption adjusts efficiently, while the emission level is held fixed at $z$. In particular, starting from the optimum $\langle \boldsymbol{z}, \boldsymbol{c}\rangle$, $dS(\boldsymbol{z}, y)$ and $dR(\boldsymbol{z}, y)$ are, respectively, the changes in the MAC and the MWTP due to the EHFE induced by a change in the level of the resource base.

Since we are assuming that welfare maximisation always results in an interior optimum, from Theorem 1, it follows that the MAC is equal to the MWTP at $\langle \boldsymbol{z}, \boldsymbol{c}\rangle$, *i.e.,* $r(\boldsymbol{z}, \boldsymbol{c}) - s(\boldsymbol{z}, \boldsymbol{c}, y) = 0$. This is true at point $A$ in Figure 1. Hence, if $dR(\boldsymbol{z}, y) - dS(\boldsymbol{z}, y) < 0$, then the EHFE results in an outcome where the MWTP for emission reduction is less than the MAC of reducing emission. This is true at point $B$ of Figure 1. It is intuitive that this difference in the consumption and production shadow prices of emission implies that the optimal emission policy requires an increase in the level of emission. This effect, which is due to the differences in the production and consumption shadow prices of emission at the outcome teached by the EHFE, is what we what we have called the DISPE above. In Figure 1, the DISPE implies a movement from point $B$ to point $C$. Similarly, if $dR(\boldsymbol{z}, y) - dS(\boldsymbol{z}, y) > 0$, then the MWTP for emission reduction is more than the MAC of reducing emission at the outcome reached by the EHFE, and the DISPE requires a fall in the level of emission. Theorem 2 states the necessary and sufficient conditions for an EKC.

**Theorem 2:** *The function $\varphi$ exhibits an inverted U-shaped EKC if and only if there exists $\overset{*}{y} > 0$ such that*

*(i) for all $0 < y < \overset{*}{y}$,*

$$dR(\boldsymbol{z}, y) < dS(\boldsymbol{z}, y), \tag{5.6}$$

*(ii) for all $y > \overset{*}{y}$,*

$$dR(\boldsymbol{z}, y) > dS(\boldsymbol{z}, y), \quad \textit{and} \tag{5.7}$$

*(iii) for $y = \overset{*}{y}$,*

$$dR(\boldsymbol{z}, y) = dS(\boldsymbol{z}, y), \tag{5.8}$$

*where $\boldsymbol{z} := \hat{z}(y)$.*

**Proof:** Define $\boldsymbol{c} := \hat{c}(y)$. Then (5.3) implies that $\boldsymbol{c} = c^m(\boldsymbol{z}, y)$. Hence, from the definitions of $dR$ and $dS$, it follows that

$$dS(\boldsymbol{z}, y) := \frac{\partial s(\boldsymbol{z}, \boldsymbol{c}, y)}{\partial c}\frac{\partial c^m(\boldsymbol{z}, y)}{\partial y} + \frac{\partial s(\boldsymbol{z}, \boldsymbol{c}, y)}{\partial y} \quad \text{and} \quad dR(\boldsymbol{z}, y) := \frac{\partial r(\boldsymbol{z}, \boldsymbol{c})}{\partial c}\frac{\partial c^m(\boldsymbol{z}, y)}{\partial y}. \tag{5.9}$$

Hence,

$$dR(\boldsymbol{z}, y) - dS(\boldsymbol{z}, y) = \left[\frac{\partial r(\boldsymbol{z}, \boldsymbol{c})}{\partial c} - \frac{\partial s(\boldsymbol{z}, \boldsymbol{c}, y)}{\partial c}\right]\frac{\partial c^m(\boldsymbol{z}, y)}{\partial y} - \frac{\partial s(\boldsymbol{z}, \boldsymbol{c}, y)}{\partial y} \tag{5.10}$$

For all $y > 0$, employing (5.3) define

$$\begin{aligned} \mathcal{R}(y) :=& r(\hat{z}(y), \hat{c}(y)) = r\left(\hat{z}(y), c^m(\hat{z}(y), y)\right) \quad \text{and} \\ \mathcal{S}(y) :=& s(\hat{z}(y), \hat{c}(y), y) = s\left(\hat{z}(y), c^m(\hat{z}(y), y), y\right). \end{aligned} \tag{5.11}$$

Theorem 1 implies that for all $y > 0$, we have $\mathcal{R}(y) \equiv \mathcal{S}(y)$. Hence, it must be the case that, for all $y > 0$

$$\frac{d\mathcal{R}(y)}{dy} - \frac{d\mathcal{S}(y)}{dy} = 0. \tag{5.12}$$

Expanding (5.12) by employing (5.11), we obtain

$$\left[\frac{\partial r(\boldsymbol{z},\boldsymbol{c})}{\partial z} + \frac{\partial r(\boldsymbol{z},\boldsymbol{c})}{\partial c}\frac{\partial c^m(\boldsymbol{z},y)}{\partial z}\right]\frac{d\hat{z}(y)}{dy} + \frac{\partial r(\boldsymbol{z},\boldsymbol{c})}{\partial c}\frac{\partial c^m(\boldsymbol{z},y)}{\partial y}$$
$$= \left[\frac{\partial s(\boldsymbol{z},\boldsymbol{c},k)}{\partial z} + \frac{\partial s(\boldsymbol{z},\boldsymbol{c},y)}{\partial c}\frac{\partial c^m(\boldsymbol{z},y)}{\partial z}\right]\frac{d\hat{z}(y)}{dy} + \frac{\partial s(\boldsymbol{z},\boldsymbol{c},y)}{\partial c}\frac{\partial c^m(\boldsymbol{z},y)}{\partial y} + \frac{\partial s(\boldsymbol{z},\boldsymbol{c},y)}{\partial y}.$$

Employing (5.9), the above can be rewritten as

$$\left[\left(\frac{\partial r(\boldsymbol{z},\boldsymbol{c})}{\partial z} + \frac{\partial r(\boldsymbol{z},\boldsymbol{c})}{\partial c}\frac{\partial c^m(\boldsymbol{z},y)}{\partial z}\right) - \left(\frac{\partial s(\boldsymbol{z},\boldsymbol{c},y)}{\partial z} + \frac{\partial s(\boldsymbol{z},\boldsymbol{c},y)}{\partial c}\frac{\partial c^m(\boldsymbol{z},y)}{\partial z}\right)\right]\frac{d\hat{z}(y)}{dy} + dR(\boldsymbol{z},y) - dS(\boldsymbol{z},y) = 0. \tag{5.13}$$

Employing (5.1) and Theorem 1, we obtain $\frac{\partial c^m(\boldsymbol{z},y)}{\partial z} = -\frac{F_z(\boldsymbol{z},\boldsymbol{c},y)}{F_c(\boldsymbol{z},\boldsymbol{c},y)} = -\frac{u_z(\boldsymbol{z},\boldsymbol{c})}{u_c(\boldsymbol{z},\boldsymbol{c})}$. Hence (5.13) implies (suppressing arguments)

$$\left[\left(\frac{\partial r}{\partial z} + \frac{\partial r}{\partial c}\left(-\frac{u_z}{u_c}\right)\right) - \left(\frac{\partial s}{\partial z} + \frac{\partial s}{\partial c}\left(-\frac{F_z}{F_y}\right)\right)\right]\frac{d\hat{z}(y)}{dy} + dR - dS = 0. \tag{5.14}$$

This implies

$$\left[\frac{1}{u_c}\left(u_c\frac{\partial r}{\partial z} - u_z\frac{\partial r}{\partial c}\right) - \frac{1}{F_c}\left(F_c\frac{\partial s}{\partial z} - F_z\frac{\partial s}{\partial c}\right)\right]\frac{d\hat{z}(y)}{dy} + dR - dS = 0. \tag{5.15}$$

Hence, employing (2.2) and (2.5), (5.15) yields

$$\left[\frac{1}{u_c^3}Q^u - \frac{1}{F_c^3}Q^F\right]\frac{d\hat{z}(y)}{dy} + dR - dS = 0. \tag{5.16}$$

Assumptions 1 and 2 imply that $\left[\frac{1}{u_c^3}Q^u - \frac{1}{F_c^3}Q^F\right] > 0$. Hence, $\frac{d\hat{z}(y)}{dy} > 0$ if and only if (5.6) is true; $\frac{d\hat{z}(y)}{dy} < 0$ if and only if (5.7) is true; and $\frac{d\hat{z}(y)}{dy} = 0$ if and only if (5.8) is true.

This implies that, if $\varphi$ exhibits an EKC, then there exists $\overset{*}{y} > 0$ such that (i), (ii), and (iii) of this theorem hold. This also implies that, if there exists $\overset{*}{y} > 0$ such that (i), (ii), and (iii) in this theorem hold, then $\varphi$ exhibits an EKC. ■

Theorem 2 clearly rules out some cases of preference and technology combinations from generating an EKC, *e.g.,* it can be inferred from this result and its proof that combinations where $dR$ is globally negative (MWTP is globally falling with increase in $y$) and $dS$ is globally positive (MAC is globally increasing with increase in $y$) result always in an upward sloping relation between optimal emission level and the economic resource base. Similarly, if $dR$ is globally positive and $dS$ is globally negative, then $\hat{z}$ is decreasing.

In Sections 6 and 7, we will employ the above characterisation of an EKC to construct and study some examples that exhibit this phenomenon in the static and the dynamic cases, when emission is a production externality caused by the use of fossil fuels. In these examples, the MAC increases when emission is held fixed and $y$ increases, *i.e.,* $dS$ is globally positive. In all the examples we assume that preferences are homothetic in the space of environmental quality and consumption. In Murty [2014] we demonstrate that this definition of homotheticity implies that the MWTP is increasing in the level of consumption, *i.e.,* $\frac{\partial r}{\partial c} > 0$. Hence, if $c^m$ is increasing in $y$, then (5.5) implies that $dR$ is also positive. Hence, the sign of $dR - dS$ is ambiguous and there there could be a scope for generating an EKC. Indeed, by a process of trial and error and some experimentation, we hit upon some parameter values that generated EKCs in the examples below.

## 6. A numerical example of an EKC: The static case.

We will follow Plassman and Khanna [2006] to model the preferences, where it is assumed that there is a bound on the maximum feasible environmental quality $\omega$, and the observed environmental quality $q$ deteriorates with increase in emission: $q = \omega - z$. Preferences over environmental quality and consumption are defined by a utility function $\bar{u} : [0, \omega] \times \mathbf{R}_+ \longrightarrow \mathbf{R}$ with image $\bar{u}(q, c)$. We will assume that $\bar{u}$ is strictly quasi-concave and twice continuously differentiable with $\bar{u}_q > 0$ and $\bar{u}_c > 0$. Given $\bar{u}$, we can derive preferences over emission and consumption, which are, here, represented by function $u : [0, \omega] \times \mathbf{R}_+ \longrightarrow \mathbf{R}$ defined as $u(z, c) := \bar{u}(\omega - z, c)$. The properties of $\bar{u}$ imply that $u$ satisfies Assumption 1. We assume that $\bar{u}$ is homothetic and has the form $\bar{u}(q, c) = (c^\eta + q^\eta)^{\frac{1}{\eta}}$. Hence,

$$u(z, c) = (c^\eta + (\omega - z)^\eta)^{\frac{1}{\eta}}, \quad \eta \leq 1, \quad \text{and} \quad \eta \neq 0. \tag{6.1}$$

In the static case, $y$ is interpreted as capital. In this numerical example, we assume $n = 1$. The intended production technology and nature's emission generating technology are specified by functions with forms

$$f(k, x) = k^\alpha x^{1-\alpha}, \quad G(z, x, a) = x^\gamma (a + \mu)^{1-\gamma} - z, \quad \alpha \in (0, 1), \ \gamma > 1, \ \text{and} \ \mu > 0. \tag{6.2}$$

We show that there exist values for the parameters that will generate an EKC.[20] We claim that the following values of the parameters serve the purpose.

$$\eta = -0.5, \ \gamma = 2, \ \mu = 2, \ \alpha = .5, \ w = 1, \ \omega = 3,$$

[20] Note, all examples in this paper are only illustrative of the theory developed in the previous sections and the values of parameters adopted are not calibrated to any macro data set.

The contours of function $c^m$ obtained from solving Problem (3.2) are shown in Panel (i) of Figure 2 in the space of capital and emission.[21] The upper contour sets of $c^m$ are convex, indicating that function $c^m$ is quasi-concave so that $F(z, c, k) := c - c^m(z, k)$ is quasi-convex as required by Assumption 2.

Function $dS$ is defined in (5.5). The contours of this function are plotted in the space of capital and emission in Panel (ii) of Figure 2. These contours indicate that, in this example, globally, $dS > 0$.

The indifference curves of function $u$ are shown in Panel (iii) of Figure 2. In this example, where $z \in [0, \omega]$, we have $\frac{\partial r}{\partial c} = c^{-\eta}(1-\eta)(\omega - z)^{-1+\eta} > 0$. From the envelope theorem applied to Problem (3.2) and (5.1), it follows that $f_k = \frac{\partial c^m}{\partial k} = -\frac{F_k}{F_c} > 0$. Hence, (5.5) implies that, globally, $dR \equiv \frac{\partial r}{\partial c}\frac{\partial c^m}{\partial k} > 0$. The contours of function $dR$ are plotted in Panel (iv) of Figure 2.

Hence, the sign of $dR - dS$ is, as such, ambiguous and an EKC is not ruled out. In particular, we find that, under our specifications of parameter values, the necessary and sufficient conditions for an EKC identified in Theorem 2 hold. Define the sets

$$\Delta := \left\{\langle k, z\rangle \in \mathbf{R}^2_{++} \;\middle|\; dR(z,k) - dS(z,k) = 0\right\} \text{ and}$$

$$Z := \left\{\langle k, z\rangle \in \mathbf{R}^2_{++} \;\middle|\; r(z, c^m(z,k)) = s(z, c^m(z,k), k)\right\}.$$

It is clear from Theorem 1 and (5.3) that the set $Z$ is the graph of function $\hat{z}(k)$. Panel (vi) of Figure 2 plots both the sets $\Delta$ (the red curve) and $Z$ (the blue curve). The blue curve clearly shows that the graph of $\hat{z}$ has an inverted-U shape, *i.e.,* this numerical example results in an EKC. Panel (v) of Figure 2 plots the contours of $dR - dS$. Panels (v) and (vi) of this figure show that points above the red curve in Panel (vi) are those where $dR - dS > 0$ and those below the red curve are those where $dR - dS < 0$. Panel (vi) of Figure 2 illustrates Theorem 2 on the necessary and sufficient conditions for an EKC. For points on the graph of $\hat{z}$ which lie below (respectively, above) the red curve, we have $dR - dS < 0$ (respectively, $dR - dS > 0$), *i.e.,* the EHFE of an increase in capital results in an outcome where the MWTP is less (respectively, more) than the MAC. Clearly, the optimal emission policy due to the DISPE implies an increase (respectively, decrease) in the level of emission. The turning point of the graph of $\hat{z}$ happens when the red and the blue curves intersect.

## 7. EKC as an outcome of economic growth.

In this section we show that the optimal trajectory of emission derived from a growth model

[21] Mathematica was employed for performing the computations and for drawing all the graphs in this section.

solves the static optimisation problem (4.1) at every time point $t$.[22] Hence, conclusions of Theorem 2 become relevant for determining whether or not the optimal trajectory of emission exhibits an EKC. This theorem provides helpful insights for constructing two numerical examples that result in an EKC.

*7.1. A property of the optimal trajectories of emission and consumption.*

Consider the growth problem:[23]

$$\begin{aligned}
&\max \int_0^\infty e^{-\rho t} u(z(t), c(t)) dt \\
&\text{subject to} \\
&\dot{k}(t) = f(k(t), x(t)) - \delta k(t) - c(t) - a(t) - wx(t), \quad a(t) = \Phi(z(t), x(t)), \\
&k(t) \geq 0,\ x(t) \geq 0,\ c(t) \geq 0,\ a(t) \geq 0,\ z(t) \geq 0,\ k(0) = k_0.
\end{aligned} \tag{7.1}$$

The current value Hamiltonian of Problem (7.1) is

$$H^c(z, c, x, a, \lambda, k) = u(z, c) + \lambda \left[ f(k, x) - \delta k - c - a - wx \right] \text{ where } a = \Phi(z, x).$$

The Hamiltonian optimised over the control variables is obtained as

$$\begin{aligned}
h^c(\lambda, k) &:= \max_{z,c,x,a} \left\{ u(z, c) + \lambda \left[ f(k, x) - \delta k - c - a - wx \right] \;\middle|\; a = \Phi(z, x) \right\} \\
&= \max_{z,c,x,a} \left\{ u(z, c) + \lambda \left[ f(k, x) - c - a - wx \right] \;\middle|\; a = \Phi(z, x) \right\} - \lambda \delta k \\
&= v(\lambda, k) - \lambda \delta k, \text{ where}
\end{aligned} \tag{7.2}$$

$$v(\lambda, k) := \max_{z,c,x,a} \left\{ u(z, c) + \lambda \left[ f(k, x) - c - a - wx \right] \;\middle|\; a = \Phi(z, x) \right\} \tag{7.3}$$

If $\hat{\hat{z}}(\lambda, k)$, $\hat{\hat{c}}(\lambda, k)$, $\hat{\hat{x}}(\lambda, k)$, and $\hat{\hat{a}}(\lambda, k)$ solve Problem (7.3), then envelope theorem yields

$$\begin{aligned}
v_\lambda(\lambda, k) &= f(k, \hat{\hat{x}}(\lambda, k)) - \hat{\hat{c}}(\lambda, k) - \hat{\hat{a}}(\lambda, k) - w\hat{\hat{x}}(\lambda, k), \\
v_k(\lambda, k) &= \lambda f_k(k, \hat{\hat{x}}(\lambda, k)).
\end{aligned}$$

The remaining first-order necessary conditions of Problem (7.1) are obtained in a standard fashion as

$$\begin{aligned}
h^c_\lambda(\lambda, k) = \dot{k} &\Longleftrightarrow \dot{k} = v_\lambda(\lambda, k) - \delta k \\
h^c_k(k, \lambda) = -\dot{\lambda} + \rho\lambda &\Longleftrightarrow \dot{\lambda} = \lambda \left[ \rho + \delta \right] - v_k(\lambda, k) \\
\lim_{t \longrightarrow \infty} e^{-\rho t} h^c &\left( k(t), \lambda(t) \right) = 0
\end{aligned} \tag{7.4}$$

[22] Recall, in the dynamic case, we interpret $y$ as time $t$.

[23] Recall (3.1) for the definition of function $\Phi$. $\rho \in \mathbf{R}_+$ denotes the rate of time preference.

The optimal trajectories of the state and co-state variables are obtained by solving the differential equation system (7.4), given the initial value of the state variable denoted by $k_0$. Suppose the optimal trajectories of the state and the co-state variables are denoted by $k^o(t, k_0)$ and $\lambda^o(t, k_0)$, respectively. Then the optimal trajectories of the control variables are obtained as solutions to (7.3), namely, $z^o(t, k_0) = \hat{\hat{z}}\left(\lambda^o(t, k_0), k^o(t, k_0)\right)$, $c^o(t, k_0) = \hat{\hat{c}}\left(\lambda^o(t, k_0), k^o(t, k_0)\right)$, $a^o(t, k_0) = \hat{\hat{a}}\left(\lambda^o(t, k_0), k^o(t, k_0)\right)$, and $x^o(t, k_0) = \hat{\hat{x}}\left(\lambda^o(t, k_0), k^o(t, k_0)\right)$.

Lemma 1, whose proof can be found in the appendix, shows that the optimal emission and consumption levels at any time point $t$ also solve the static optimisation problem (4.1). Theorem 3 follows in an obvious manner from Lemma 1.

**Lemma 1:** *Suppose the trajectory of capital is given by $k^o(t, k_0)$. Suppose function $c^m$ is obtained by solving the Problem (3.5), function $F$ is defined as in (3.7), and the set $Y(t)$ is defined as in (3.6) for all $t \geq 0$. Then, for any $t \geq 0$, $\langle z^o(t, k_0),\ c^o(t, k_0)\rangle$ solves the static optimisation problem (4.1) where $y$ is interpreted as time $t$, i.e., $z^o(t, k_0) = \hat{z}(t)$ for all $t \geq 0$.*

**Theorem 3:** *Under the assumptions of Lemma 1, the graph of $z^o(t, k_0)$ has an inverted-U shape if $\varphi(t) = \langle \hat{z}(t),\ \hat{c}(t)\rangle$ exhibits an EKC.*

Thus, Theorem 3 implies that the optimal trajectory of emission $z^o(t, k_0)$ has an inverted-U shape if the conditions in Theorem 2 hold. Conditions in Theorem 2 are in terms of the difference $dR - dS$. The set of feasible emission-consumption combinations was derived in Section 3.3 for the dynamic case. At any time $t$, this set depends on the availability of capital stock at $t$. If (3.1) is also true then, under Assumption 4, $\Phi_z = -\frac{G_z}{G_a} < 0$ and $\Phi_{x_j} = -\frac{G_{x_j}}{G_a} > 0$ for all $j = 1, \ldots, n$. Theorem 4, below, derives the expressions for functions $dS$ and $dR$ in the dynamic case. Its proof is relegated to the appendix.

**Theorem 4:** *Under Assumptions 1, 3, and 4, we have*

$$dS(z,t) = \nabla_{zx}\Phi(z,\boldsymbol{x}) \cdot \left[\nabla_{xx} f(k(t),\boldsymbol{x}) - \nabla_{xx}\Phi(z,\boldsymbol{x})\right]^{-1} \nabla_{xk} f(k(t),\boldsymbol{x})\dot{k}(t) \quad \textit{and}$$
$$dR(z,t) = \left[-\left(\frac{(\omega - z)\ u_c(z,\mathbf{c})}{-u_c(z,\mathbf{c})\mathbf{c} + u_z(z,\mathbf{c})(\omega - z)}\right)\frac{Q^u(z,\mathbf{c})}{u_c^3(z,\mathbf{c})}\right]\left[\left(f_k(k(t),\boldsymbol{x}) - \delta\right)\dot{k}(t) - \frac{d\dot{k}(t)}{dt}\right], \tag{7.5}$$

*where $\mathbf{x} = \tilde{x}(z,t)$ and $\mathbf{c} = c^m(z,t)$.*[24]

[24] $\nabla_x\Phi$ is the gradient of function $\Phi$ with respect to $x$, while $\nabla_{zx}\Phi$ and $\nabla_{xx}\Phi$ are the matrices of second order derivatives of $\Phi$.

The numerical examples presented in the next two sections result in EKCs. In both sections, preferences are homothetic and are defined as in (6.1). In Murty [2014] we have shown that, if preferences are homothetic in the space of environmental quality and consumption, then $\left[-\left(\frac{(\omega-z)\ u_c(z,\mathbf{c})}{-u_c(z,\mathbf{c})\mathbf{c}+u_z(z,\mathbf{c})(\omega-z)}\right)\frac{Q^u(z,\mathbf{c})}{u_c^3(z,\mathbf{c})}\right] = \frac{\partial r(z,c)}{\partial c} > 0$ for all $\langle z, c\rangle \in \mathbf{R}^2_{++}$. Hence, from (5.5) it follows that the sign of $dR(\boldsymbol{z}, t)$ depends on the the sign of the term $\frac{\partial c^m}{\partial t}$. Applying envelope theorem to Problem (3.5), this latter term is $(f_k(k^o(t,k_0),\boldsymbol{x}) - \delta)\,\dot{k}^o(t,k_0) - \frac{d\dot{k}^o(t,k_0)}{dt}$.

*7.2. A numerical example with $n = 1$.*

Consider the case where there is only one type of fossil fuel, *i.e.,* $n = 1$. Suppose the technological specifications are

$$f(k,x) = Ak + k^{\alpha}x^{1-\alpha}, \qquad \Phi(z,x) = \left(\frac{x^{\gamma}}{z}\right)^{\frac{1}{\gamma-1}} - \mu, \qquad \gamma \geq 1,\ \mu > 0,\ A > 0,\ \alpha \in (0,1). \tag{7.6}$$

In this case, Theorem 4 implies that, along the trajectory of capital $k^o(t,k_0)$ that solves Problem (7.1), we have

$$dS(z,t) = \left(\frac{\Phi_{zx}\,(z,\boldsymbol{x})\,f_{xk}\,(k^o(t,k_0),\boldsymbol{x})}{f_{xx}\,(k^o(t,k_0)),\boldsymbol{x}) - \Phi_{xx}\,(z,\boldsymbol{x})}\right)\dot{k}^o(t,k_0) \quad \text{where} \quad \boldsymbol{x} = \tilde{x}(z, k^o(t,k_0)). \tag{7.7}$$

It can be verified that, under the functional forms adopted in (7.6), $f_{xk} > 0$, $f_{xx} < 0$, $\Phi_{xx} \geq 0$, and $\Phi_{xz} < 0$. Hence, if the optimal trajectory of capital $k^o(t,k_0)$ is increasing over time, then $dS(\boldsymbol{z},t) > 0$ for all $t$, where $\boldsymbol{z} = z^o(t,k_0)$.

The functional form of $f$ in (7.6) is reminiscent of the form considered by Jones and Manuelli [1990], which led to endogenous growth. It has an $Ak$ part and a neo-classical part. But, the difference is that, in Jones and Manuelli, the inputs are labour and capital and labour grows exogenously, while in our context the inputs are fossil fuel and capital and the growth of fossil-fuel is endogenous to the model. We solve the problem by adopting a numerical approach.[25] Recall, under our specification of preferences, the emission level lies in the bounded interval $[0,\omega]$. As is normal in numerical analyses, a grid indicating the range of values that capital can take needs to be specified. In addition, we found that we also had to specify a range of values that the fossil fuel level can take. This is because values of the other two control variables follow once the values of capital, emission level, and fossil fuel level are know.[26] In fact, our numerical analysis can be thought of as solving a dynamic programming problem with the state variable

[25] Matlab was employed for the numerical analyses undertaken in this section.

[26] The level of abatement is determined by function $\Phi$ once the levels of the fossil-fuel and emission are known. The level of consumption is known from $f(k,x) - \dot{k} - \delta k - wx - \Phi(z,x)$ once the levels of capital, investment, fossil fuel, and abatement are known.

capital and the control variable fossil fuel usage being bounded. We found that the results do not qualitatively change when the bounds on capital and the fossil-fuel are changed. The following values for the parameters were specified:

$$\rho = 0.05\%, \quad \alpha = 0.8, \quad \delta = 10^{-6}, \quad \gamma = 2, \quad \mu = 2, \quad \eta = 0.5, \quad w = 0.001, \quad \omega = 3.$$

The optimal trajectories resulting from the numerical analysis for an initial value of capital equal to 51 are shown in Figure 3. Capital stock increases over time and hits its upper bound, which, in this numerical example, we have specified as 1000. We found that this holds true even when we change the upper bound arbitrarily. The optimal trajectory for capital suggests endogenous growth in our model, where function $f$ has the Jones and Manuelli [1990] form. Hence, since capital grows over time, (7.7) implies that $dS(z,t) > 0$ in this example.

Figure 3 reveals that the trajectory of emission has an inverted-U shape: emission level first rises and reaches a peak and then falls. This figure also shows that both the fossil-fuel usage and the level of abatement are also increasing over time.[27] This implies that, in the rising part of the emission trajectory, the increase in emission level due to increase in the usage of the fossil fuel offsets the increase in emission mitigation due to the increase in the abatement effort. The converse must be true in the declining part of the emission trajectory. Since $dS(z,t) > 0$ in this example and the model exhibits an EKC, Theorem 2 implies that, in the declining part of the emission trajectory, it must be the case that $dR(z,t) > 0$, *i.e.,* if an EKC is observed in a situation where the EHFE implies increasing MAC throughout, then it must imply increasing MWTP in the falling part of the optimal emission trajectory. Hence, Theorem 4 implies that $\frac{\partial c^m}{\partial t} = \left(f_k(k^o(t,k_0),\boldsymbol{x}) - \delta\right)\dot{k}^o(t,k_0) - \frac{d\dot{k}^o(t,k_0)}{dt} > 0$ in this region.

*7.3. EKC with inter-fuel substitution possibilities.*

Here, we assume two energy (fuel) inputs, *i.e.,* $n = 2$. They differ in terms of their costs of extraction and in their emission-generating capacities. This motivates the possibility of inter-fuel substitution during the growth process. Suppose the function $\Psi : \mathbf{R}_+^2 \longrightarrow \mathbf{R}_+$ with image $\Psi(x_1, x_2)$ measures the energy generated by the two fossil fuels. The technological and cost specifications are extensions of those in (7.6):

$$f(k,x) = Ak + k^\alpha \Psi(x_1,x_2)^{1-\alpha}, \;\; \Phi(z,x) = \left(\frac{(\gamma_1 x_1 + \gamma_2 x_2)^\gamma}{z}\right)^{\frac{1}{\gamma-1}} - \mu, \quad \Psi(x_1,x_2) = (x_1^\chi + x_2^\chi)^{\frac{1}{\chi}}$$
$$\gamma \geq 1,\; \gamma_1 > \gamma_2 \geq 0,\; \mu > 0,\; A > 0,\; \alpha \in (0,1),\; \chi \leq 1,\; \chi \neq 0,\; w_2 > w_1 \geq 0. \tag{7.8}$$

[27] But the level of the fossil fuel is always below the upper bound (which is 10) that we have specified on the range of values it can take.

(7.8) implies that the first fuel input is less costly (as $w_1 < w_2$) but also more dirty, *i.e.,* has a greater emission-generating intensity. The latter is because (3.1) implies $z = \Theta(x,a) = (\gamma_1 x_1 + \gamma_2 x_2)^{\gamma} (a + \mu)^{1-\gamma}$, so that the emission intensity of the $j^{th}$ fuel input for $j = 1,2$ (which is defined as marginal emission generated by the $j^{th}$ fuel input) is $\frac{\partial\Theta(x,a)}{\partial x_j} = (a + \mu)^{1-\gamma}\gamma\,(\gamma_1 x_1 + \gamma_2 x_2)^{\gamma-1}\gamma_j$. Since $\gamma_1 > \gamma_2$, it is clear that the first input is the more dirty input as its use leads to higher incremental emission generation. The following values for the parameters were specified:

$$\rho = 0.05\%, \quad \alpha = 0.8, \;\; \delta = 10^{-6}, \quad \gamma = 2, \quad \gamma_1 = 2, \quad \gamma_2 = 1, \quad \mu = 2, \quad \eta = 0.5,$$
$$w_1 = 0.001, \quad w_2 = 1, \;\; \chi = 0.5, \quad \omega = 3.$$

A numerical analysis of this model is conducted under bounded domains for capital and the two fossil-fuels. The resulting trajectories of the state and control variables are presented in Figure 4 for an initial value of capital equal to 151. Panel (i) of this figure shows that capital grows over time and hits the upper bound that has been imposed by us (which is 2000). This remains true for other values of the upper bound as well suggesting endogenous growth of capital. The usage of the cheaper but dirty fossil fuel increases at first and then falls, while the usage of the more expensive and cleaner fossil fuel increases throughout. Both fuels take values less than the upper bounds we have specified for their domain (which is 20). The level of abatement increases at first, then falls for a while, and then rises again. The trajectory of emission exhibits an EKC. The rising part of this trajectory is characterised by rising usage of both fossil-fuels and rising abatement effort. Hence, the latter's mitigating effect on emission generation must be dominated by the expansionary effect the former have on emission generation. The falling part of the emission trajectory is characterised by inter-fuel substitution (increasing use of the cleaner fuel input and reduction in the use of the dirtier fossil-fuel input) as well as a mixed trend in the abatement effort. As in the previous examples, in this example also, the EHFE results in rising MAC. Hence, Theorem 2 implies that the declining part of the emission trajectory corresponds to rising MWTP under the EHFE. One can perhaps conclude from this that, in this region, the increased preference for the environment relative to consumption is what motivates the increasing trends in abatement effort and inter-fuel substitution.

## 8. Decentralisation of the social optimum.

We show that, given $y \geq 0$, the decentralisation of the social optimum $\varphi(y)$ of the welfare maximisation problem (4.1) in the case where the emission is induced by fossil fuels could be achieved either by standard emission policies such as a Pigou tax or an emission standard or by a scheme of non-linear subsidisation of abatement effort and linear taxation of fossil-fuel usage. In particular, the latter scheme ensures that the firms will choose fossil-fuel usage and the level of abatement effort optimally, both of which would ensure the optimal generation of emission.

To see this, WOLOG, let us assume a static framework and interpret $y$ as capital. Let us rewrite Problem (4.1) by employing the details of the model of the technology that generates emission due to the use of fossil fuels as inputs in Section 3:

$$\max_{z,c,x,a} \left\{ u(z,c) \;\middle|\; c \leq f(k,x) - wx - a,\ G(z,x,a) \leq 0 \right\}. \tag{8.1}$$

If $\varphi(k) = \langle \boldsymbol{z}, \boldsymbol{c} \rangle$, $\boldsymbol{a}$, and $\boldsymbol{x}$ solve the above problem, then it can be verified that the first-order conditions of above maximisation yield:

$$f_{x_j}(k, \boldsymbol{x}) = w_j + \left( -\frac{G_{x_j}(\boldsymbol{z}, \boldsymbol{x}, \boldsymbol{a})}{G_a(\boldsymbol{z}, \boldsymbol{x}, \boldsymbol{a})} \right) \ \forall\, j = 1, \ldots, n, \quad -\frac{u_z(\boldsymbol{z}, \boldsymbol{c})}{u_c(\boldsymbol{z}, \boldsymbol{c})} = \frac{G_z(\boldsymbol{z}, \boldsymbol{x}, \boldsymbol{a})}{G_a(\boldsymbol{z}, \boldsymbol{x}, \boldsymbol{a})}, \qquad G(\boldsymbol{z}, \boldsymbol{x}, \boldsymbol{a}) = 0, \quad \text{and} \quad \boldsymbol{c} = f(k, \boldsymbol{x}) - w\boldsymbol{x} - \boldsymbol{a}. \tag{8.2}$$

Recall that $-\frac{G_{x_j}}{G_a}$ for all $j = 1, \ldots, n$ is the trade-off between the $j^{th}$ fossil fuel and abatement when emission is held fixed. Evaluated at the social optimum, it reflects the increase in abatement required to maintain the level of emission at $\boldsymbol{z}$ when the usage of the $j^{th}$ fossil fuel increases by a unit. This increase in abatement is a cost to the society as it reduces the amount of consumption available from the total output that is produced. The first equality in (8.2), hence, implies that, at the social optimum, the marginal benefit from an increase in this fossil fuel, given by $f_{x_j}$, is equal to the total marginal cost of this increase, which is given by $w_j + \left(-\frac{G_{x_j}}{G_a}\right)$.

Employing (5.1) and applying the envelope theorem to (3.2), we obtain $s(\boldsymbol{z}, \boldsymbol{c}, k) = \frac{\partial c^m(\boldsymbol{z}, k)}{\partial z} = \frac{G_z(\boldsymbol{z}, \boldsymbol{x}, \boldsymbol{a})}{G_a(\boldsymbol{z}, \boldsymbol{x}, \boldsymbol{a})} > 0$. The second equality in (8.2), hence, reflects the equality of the MWTP and the MAC at the optimum. This is as stated in Theorem 1.

Let's focus on the decentralisation of the socially optimal level of production as the decentralisation in the consumption sector through markets follows in a standard way, requiring the implementation of a scheme of lump-sum transfers by the government. For convenience in exposition but WOLOG, we assume a single aggregated profit maximising firm.[28] Assume that the price of the consumption good is normalised to one and that the firm faces the market prices of capital ($r > 0$) and fossil fuels ($w \in \mathbf{R}^n_+$). The firm can use the output it produces to either sell in the market as the consumption good or for abating its emission.

### *8.1. Decentralisation using a Pigou tax.*

Suppose the government employs a standard Pigou tax, which is a linear tax $\tau$, to regulate

[28] The analysis can be extended in an obvious way to the case of several firms.

the emission generated by the firm.[29] The firm solves the problem:

$$\max_{a,x,c,z,k} \left\{ c - rk - wx - \tau z \;\middle|\; a + c \leq f(k,x),\ G(z,x,a) \leq 0 \right\}. \tag{8.3}$$

It can be verified that the first, third, and the fourth equalities in (8.2) follow from the first-order conditions of Problem (8.3). In addition, the first-order conditions of (8.3) also yield:

$$f_k(k,x) = r \quad \text{and} \quad \tau = \frac{G_z(z,x,a)}{G_a(z,x,a)}.$$

The market rate of interest on capital reflects it marginal product. By setting the Pigou tax rate $\tau$ equal to $-\frac{u_z(\boldsymbol{z},\boldsymbol{c})}{u_c(\boldsymbol{z},\boldsymbol{c})}$, the solution to (8.1) can be decentralised.

*8.2. Decentralisation through nonlinear subsidisation of abatement and linear taxation of fossil fuels.*

Now consider a taxation scheme which involves (i) a non-linear subsidy schedule $\Gamma(a)$ given to the firm for undertaking abatement effort and (ii) a system of linear taxation of fossil fuels at the rates $t \in \mathbf{R}^n$. The firm now solves

$$\max_{a,x,c,z,k} \left\{ c + \Gamma(a) - rk - (w+t)x \;\middle|\; a + c \leq f(k,x),\ G(z,x,a) \leq 0 \right\}. \tag{8.4}$$

It can be verified (from the first order condition with respect to $z$) that the Lagrange multiplier of Problem (8.4) for the constraint $G(z,x,a) \leq 0$ will be zero. The first-order conditions of Problem (8.4) yield:

$$\nabla_x f(k,x) = w + t, \quad f_k(k,x) = r, \quad \Gamma'(a) = 1, \quad a + c = f(k,x). \tag{8.5}$$

Consider a non-decreasing, concave, and nonlinear subsidy schedule $\Gamma$ which ensures that, evaluated at the optimal abatement level $\boldsymbol{a}$, we have $\Gamma'(\boldsymbol{a}) = 1$. Also choose the levels of the linear tax rates on the fossil fuels as $t_j = -\frac{G_{x_j}(\boldsymbol{z},\boldsymbol{x},\boldsymbol{a})}{G_a(\boldsymbol{z},\boldsymbol{x},\boldsymbol{a})}$ for all $j = 1, \ldots, n$. Then, comparing (8.2) with (8.5), it is clear that this scheme of taxation and subsidisation will mean that the firm chooses the socially optimal levels of fossil fuels and abatement effort. This implies that the nature's emission generation technology will generate the optimal level of net emission. Hence, the solution to (8.1) can be decentralised by this scheme. An example of such a subsidy schedule is

$$\begin{aligned} \Gamma(a) &= \frac{\alpha}{2}a^2 + \beta a \quad \forall a \in \left[0, -\frac{\beta}{\alpha}\right] \\ &= -\frac{\beta^2}{2\alpha} \qquad \forall a \geq -\frac{\beta}{\alpha}, \qquad\qquad \alpha = \frac{1-\beta}{\boldsymbol{a}},\ \beta > 1 \end{aligned} \tag{8.6}$$

To see why this subsidy helps in decentralisation, for the sake of illustration, assume that the socially optimal level of abatement is $\boldsymbol{a} = 10$. Choose $\beta = 2$. Then $\alpha = -\frac{1}{10}$ and $-\frac{\beta}{\alpha} = 20$. The graph of this schedule is plotted in blue in Figure 5. At $a = \boldsymbol{a} = 10$, the slope of this schedule is $\Gamma'(\boldsymbol{a}) = \alpha\boldsymbol{a} + \beta = 1$.

[29] An emission standard can also be used as an alternative to the Pigou tax.

## 9. Conclusions.

Theorem 2 of our paper on the necessary and sufficient conditions for an EKC formalises our intuition regarding the occurrence of this phenomenon, namely, as countries transition from subsistence to affluent states due to the growth in their economic resource base, their valuation for environment relative to material well-being may change and /or the technological costs of abating emission may change. This theorem states that an EKC arises if and only if there is a threshold level of resource-base such that, at all levels of the resource-base below (respectively, above) this threshold, the MAC increases (respectively, decreases) more than the MWTP when the resource base increases but the emission policy is not allowed to adjust. This implies that the optimal emission policy following an increase in the resource base is one where the emission level should increase (respectively, decrease).

In this paper, we propose a model of fossil-fuel induced emission. Theorem 2 provides some insights into understanding preference and technology combinations that can potentially result in an EKC. In particular, we choose preference and technology structures where the MWTP and the MAC are both increasing in the resource base when the emission level is held fixed. For some parameter values, the relative changes in these two shadow prices of emission are in tune with the conclusions of Theorem 2 and we generate EKCs in the model with fossil-fuel induced emission in both static and dynamic frameworks. The optimal mix of emission-mitigation strategies in our examples are also revealed in our numerical analysis. In particular, in the model with multiple energy inputs with varying costs of extraction and emission generating intensities, we find that the declining part of the EKC is characterised by considerable inter-fuel substitution from the dirtier to the cleaner fossil fuel and an increase in explicit abatement effort.

A model, where emission generation is attributed directly to its causes, *e.g.,* the combustion of fossil fuels, is helpful in revealing the importance of real-world policies which aim at regulating emission based on its sources and methods of mitigation. *E.g.,* the policy of subsidising abatement activities discussed in Section 8 is reminiscent of the Clean Development Mechanism (CDM) advocated in the Kyoto Protocol, where countries can potentially receive rewards for abatement activities. Real world governmental policies also include taxation of energy inputs to correct for the externalities, such as emission, that they generate. We show that such policies could be perfect substitutes for classic emission policies such as the Pigouvian tax or emission standards.

## APPENDIX

**Proof of Lemma 1:** Suppose not. Then, there exists $\langle \bar{z}, \bar{c} \rangle \neq \langle z^o(t, k_0),\ c^o(t, k_0) \rangle$ that solves (4.1). This implies $\langle \bar{z}, \bar{c} \rangle$ is a strictly efficient point of $Y(t)$. Hence, $F(\bar{z}, \bar{c}, t) = 0$. Hence, (3.7) implies that $\bar{c} = c^m(\bar{z}, t)$. Hence, there exist $\bar{x}$ and $\bar{a}$ that solve Problem (3.5) for $t$ and $z = \bar{z}$, *i.e.*, $\bar{x} = \tilde{x}(\bar{z}, t)$ and $\bar{a} = \Phi(\bar{z}, \bar{x})$. Hence, $f(k^o(t, k_0), \bar{x}) - \delta k^o(t, k_0) - \dot{k}^o(t, k_0) - \bar{a} - w\bar{x} = \bar{c}$. Hence, we have

$$f(k^o(t,k_0),\bar{x}) - \bar{c} - \bar{a} - w\bar{x} = \delta k^o(t,k_0) + \dot{k}^o(t,k_0) = f(k^o(t,k_0), x^o(t,k_0)) - c^o(t,k_0) - a^o(t,k_0) - wx^o(t,k_0),$$

since the latter equality is true along the optimal trajectories of Problem (7.1). Hence,

$$\begin{aligned} &\lambda^o(t,k_0)\left[f(k^o(t,k_0),\bar{x}) - \bar{c} - \bar{a} - w\bar{x}\right] \\ &\quad = \lambda^o(t,k_0)\left[f(k^o(t,k_0), x^o(t,k_0)) - c^o(t,k_0) - a^o(t,k_0) - wx^o(t,k_0)\right] \end{aligned} \tag{A.1}$$

Since $a^o(t,k_0) = \Phi(z^o(t,k_0), x^o(t,k_0))$, the definition of $c^m$ in (3.5) implies $c^m(z^o(t,k_0),t) \geq f(k^o(t,k_0), x^o(t,k_0)) - \delta k^o(t,k_0) - \dot{k}^o(t,k_0) - a^o(t,k_0) - wx^o(t,k_0) = c^o(t,k_0)$. Hence, (7.5) implies $F(z^o(t,k_0), c^o(t,k_0), t) \leq 0$. Hence, $\langle z^o(t,k_0),\ c^o(t,k_0)\rangle \in Y(t)$. But since $\langle \bar{z}, \bar{c}\rangle$ solves Problem (4.1) for $t$ and $\langle z^o(t,k_0),\ c^o(t,k_0)\rangle$ does not, it must be the case that

$$u(\bar{z},\bar{c}) > u\left(z^o(t,k_0),\ c^o(t,k_0)\right) \tag{A.2}$$

(A.1) and (A.2) imply that

$$\begin{aligned} &u(\bar{z},\bar{c}) + \lambda^o(t,k_0)\left[f(k^o(t,k_0),\bar{x}) - \bar{c} - \bar{a} - w\bar{x}\right] \\ &\quad > u\left(z^o(t,k_0),\ c^o(t,k_0)\right) + \lambda^o(t,k_0)\left[f(k^o(t,k_0), x^o(t,k_0)) - c^o(t,k_0) - a^o(t,k_0) - wx^o(t,k_0)\right]. \end{aligned}$$

Since $\bar{a} = \Phi(\bar{z}, \bar{x})$ and $a^o(t,k_0) = \Phi(z^o(t,k_0), x^o(t,k_0))$, the above contradicts the fact that $\left\langle z^o(t,k_0), c^o(t,k_0), x^o(t,k_0), a^o(t,k_0)\right\rangle$ solves (7.3) for $\lambda = \lambda^o(t,k_0)$ and $k = k^o(t,k_0)$. ■

**Proof of Theorem 4:** From (5.4) and the envelope theorem applied to Problem (3.5), we have $s(z, c^m(z,t), t) = \frac{\partial c^m(z,t)}{\partial z} = -\Phi_z(z, \boldsymbol{x})$. Hence, (5.5) implies that[30]

$$\begin{aligned} dS(z,t) &= -\sum_{j=1}^{n} \Phi_{zx_j}(z, \tilde{x}(z,t)) \frac{\partial \tilde{x}_j(z,t)}{\partial t} \\ &= -\nabla_{zx}\Phi(z,\boldsymbol{x}) \cdot \nabla_t \tilde{x}(z,t). \end{aligned} \tag{A.3}$$

The first order condition of Problem (3.5) for an interior solution is

$$\nabla_x f(k(t),\boldsymbol{x}) - \nabla_x \Phi(z,\boldsymbol{x}) = w. \tag{A.4}$$

[30] $\nabla_x\Phi$ denotes the gradient of function $\Phi$ with respect to $x$, while $\nabla_{zx}\Phi$ denotes the gradient of $\frac{\partial \Phi}{\partial z}$ with respect to $x$. $\nabla_{xx}\Phi$ denotes the Hessian of $\Phi$ with respect to $x$. $\nabla_t\tilde{x}$ is the gradient of the vector valued function $\tilde{x} : \mathbf{R}^2_+ \longrightarrow \mathbf{R}^n$ with respect to $t$. Likewise, we define the matrix of derivatives of function $f$ in the proof above.

Differentiating both sides of (A.4) with respect to $t$, we obtain

$$\nabla_{xk} f(k(t), \boldsymbol{x})\dot{k}(t) + \nabla_{xx} f(k(t), \boldsymbol{x})\nabla_t \tilde{x}(z,t) - \nabla_{xx}\Phi(z, \boldsymbol{x})\nabla_t \tilde{x}(z,t) = 0^n. \tag{A.5}$$

Solving for $\nabla_t \tilde{x}(z,t)$, we obtain

$$\nabla_t \tilde{x}(z,t) = -\left[\nabla_{xx} f(k(t), \boldsymbol{x}) - \nabla_{xx}\Phi(z,t)\right]^{-1} \nabla_{xk} f(k(t), \boldsymbol{x})\dot{k}(t). \tag{A.6}$$

Plugging into (A.3), we obtain the first part of (7.5). In Murty [2014], we show that, if $\bar{u}$ is homothetic, then

$$\frac{\partial r(z, c^m(z,t))}{\partial c} = -\left(\frac{(\omega - z)\ u_c(z, \mathbf{c})}{-u_c(z, \mathbf{c})\mathbf{c} + u_z(z, \mathbf{c})(\omega - z)}\right) \frac{Q^u(z, \mathbf{c})}{u_c^3(z, \mathbf{c})}. \tag{A.7}$$

From the envelope theorem applied to Problem (3.5), we have

$$\frac{\partial c^m(z,t)}{\partial t} = \left(f_k(k(t), \boldsymbol{x}) - \delta\right)\dot{k}(t) - \frac{d\dot{k}(t)}{dt}. \tag{A.8}$$

Hence, from (5.5) it follows that together (A.7) and (A.8) imply the second part of (7.5). ■

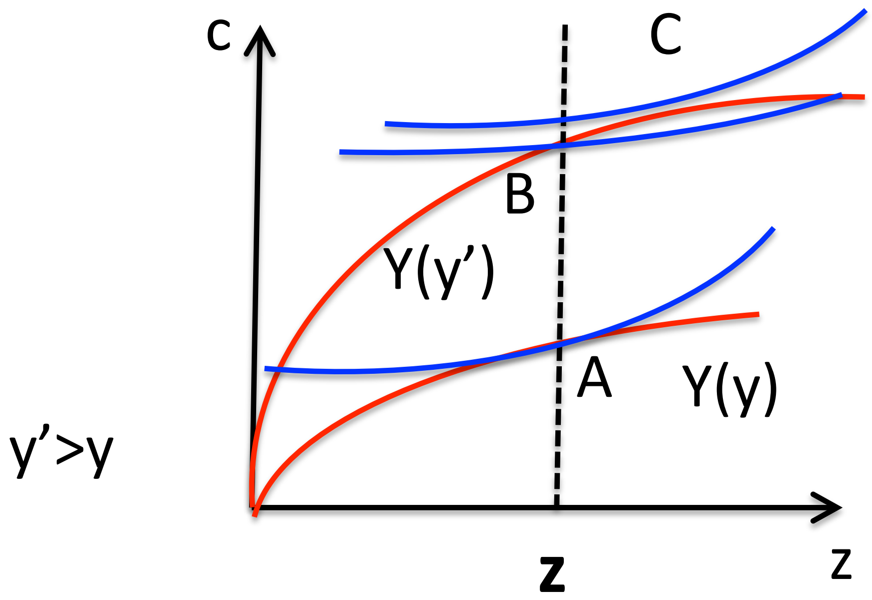
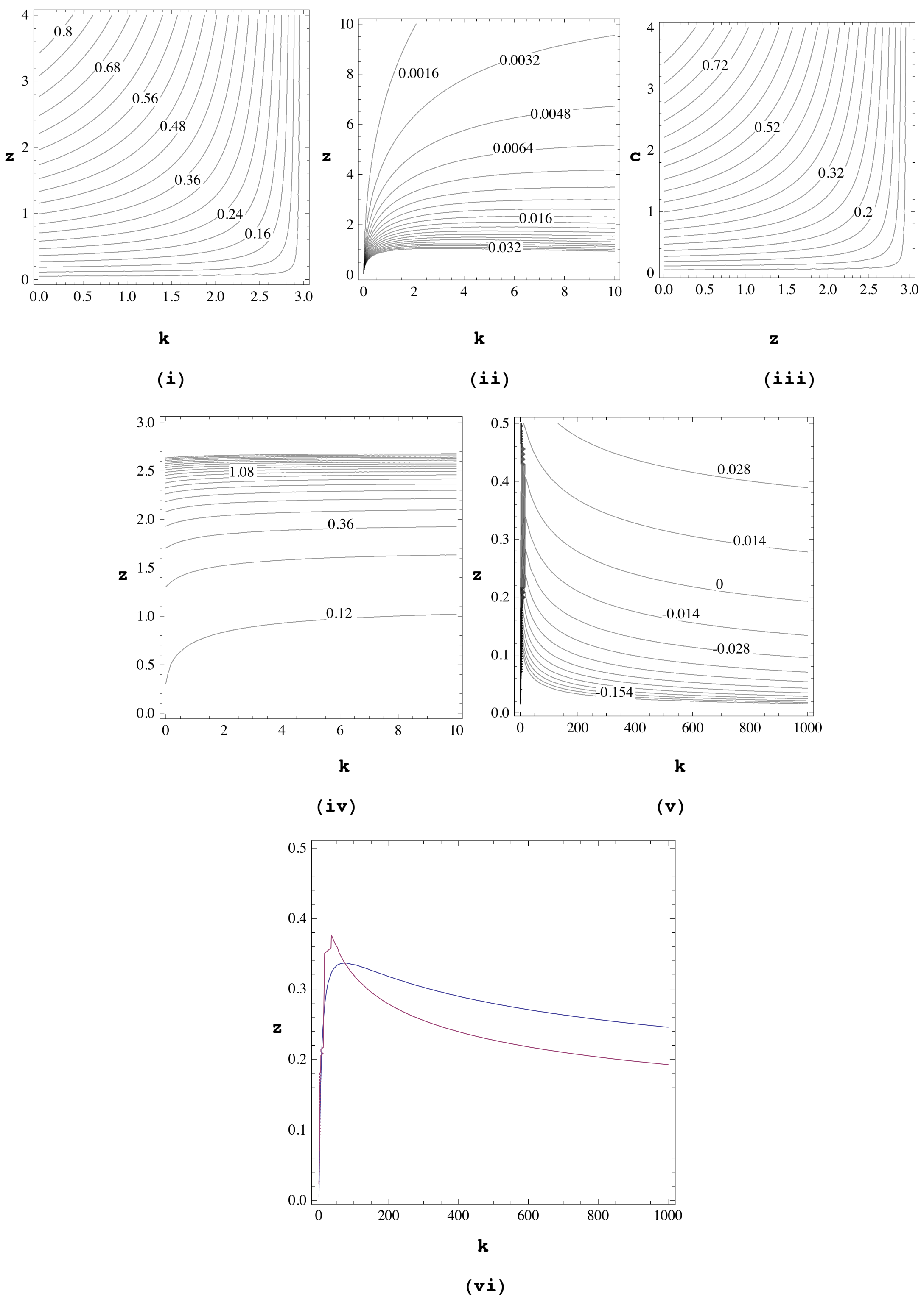



Figure 1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0.56
0.48
0.36
0.24
0.16
z
k
(i)
0.0016
0.0032
0.0048
0.0064
0.016
0.032
z
k
(ii)
0.72
0.52
0.32
0.2
c
z
(iii)
1.08
0.36
0.12
z
k
(iv)
0.028
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-0.014
-0.028
-0.154
z
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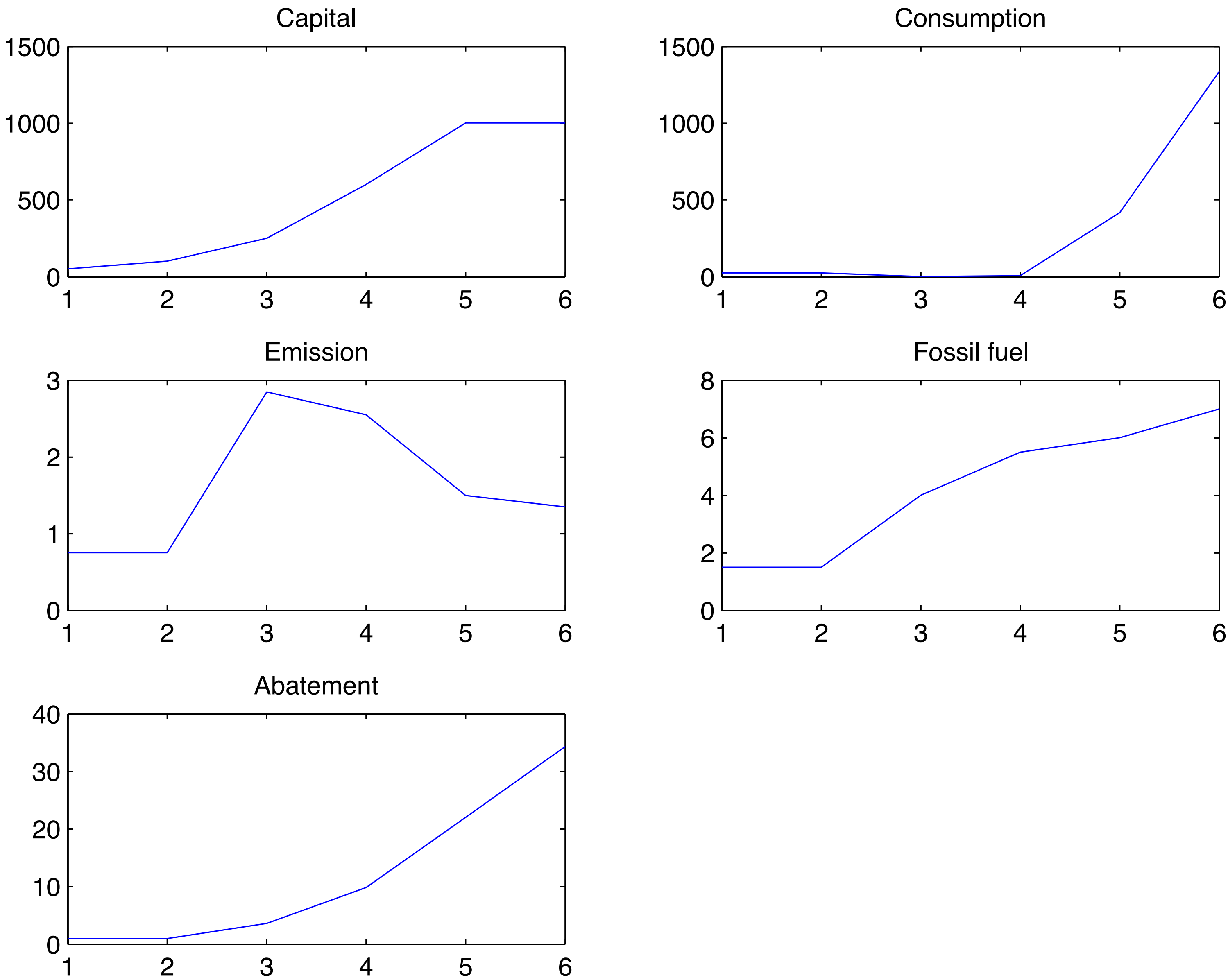
(v)
z
k
(vi)

Figure 2



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1000
500
0
1
2
3
4
5
6
Consumption
1500
1000
500
0
1
2
3
4
5
6
Emission
3
2
1
0
1
2
3
4
5
6
Fossil fuel
8
6
4
2
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3
4
5
6


Figure 3

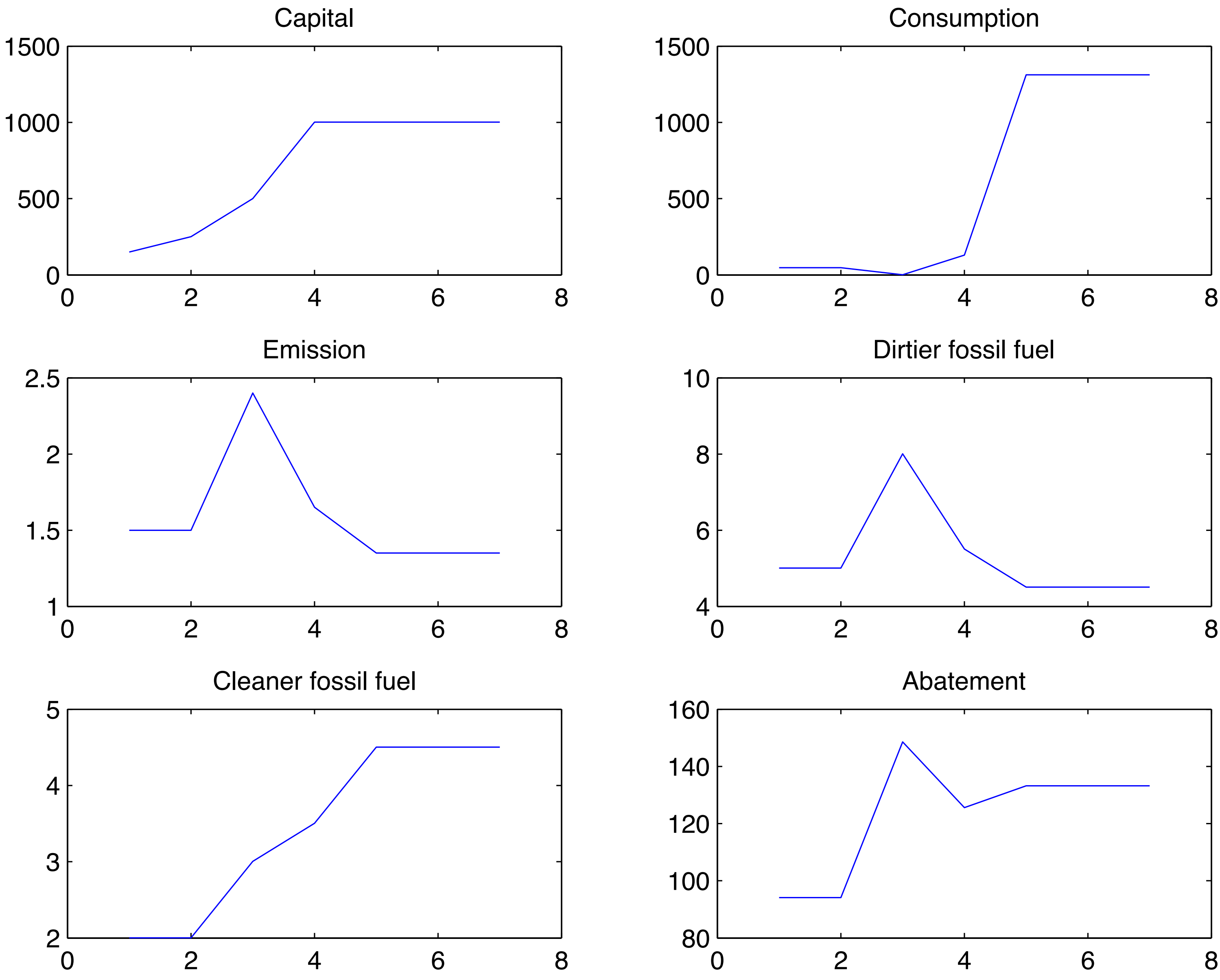
Capital
Consumption
Emission
Dirtier fossil fuel
Cleaner fossil fuel
Abatement

Figure 4

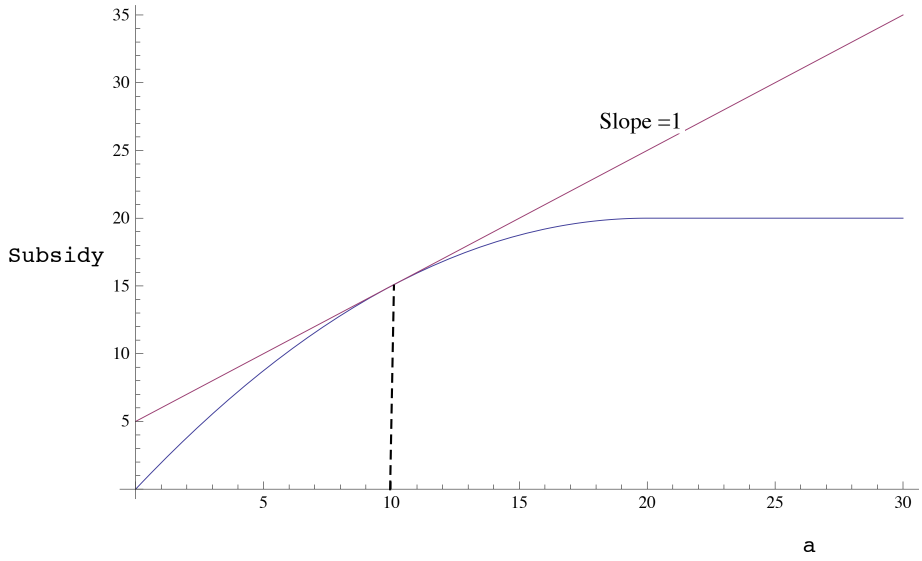


Figure 5

## REFERENCES


Andreoni, J., Levinson A., 2001. "The simple analytics of the environmental Kuznets curve," *Journal of public economics*, 80: 269–286.

Dasgupta, S., Laplante B,, Wang H., Wheeler D., 2002. "Confronting the environmental Kuznets curve," *Journal of economic perspectives* 16 (1): 147–168.

Dinda, S., 2004. "Environmental Kuznets curve hypothesis: A survey," *Ecological economics,* 49: 431–455.

Egli H., Steger T. M., 2007. " A dynamic model of the environmental Kuznets curve: Turning point and public policy," *Environment and resource economics,* 36: 15–34.

Grossman, G. M., Krueger A. B., 1993. "Environmental impacts of a North-American free trade agreement," *The U. S. - Mexico free trade agreement*, P. Garber, ed., The MIT press, Cambridge Massachusetts.

Grossman, G. M. Krueger A. B., 1995. "Economic growth and the environment," *Quarterly journal of economics*, 110: 353–377.

Holz-Eakin, D., Selden T. M., 1995. "Stoking the fires? $CO_2$ emissions and economic growth," *Journal of public economics*, 57: 85–101.

John, A., Pecchenino R., 1994. "An overlapping generations model of growth and the environment," *Economic journal*, 104: 1393–1410, 1994.

Jones, L. E., and R. E. Manuelli, "Endogenous policy choice: the case of pollution and growth," *Review of economic dynamics*, 4: 369–405, 1994.

Jones, L. E., Manuelli R. E., 1990. "A convex model of economic growth: Theory and pollcy implications," *Journal of polltical economy,* 98(5): 1008–1038, 1990.

Kijima M., Nishide K., Ohyama A., 2010. "Economic models for the environmental Kuznets curve: A survey," *Journal of economic dynamics and control*, 34: 1187–1201, 2010.

Lieb, C.M., 2002. "The environmental Kuznets curve and satiation: a simple static model," *Environment and development economics,* 7(3): 429–448, 2002.

Lopez, R., 2002. "The environment as a factor of production: the effects of economic growth and trade liberalization," *Journal of environmental economics and management*, 27: 163–184, 1994.

Lopez, R., S. Mitra S., 2000. "Corruption, pollution, and the Kuznets environment curve," *Journal of environmental economics and management*, 40 (2): 137– 150, 2000.

Mason R., Swanson T., 2003. "A Kuznets curve analysis of ozone-depleting substances and the Montreal protocol," *Oxford economic papers*, 55: 1–24.

McConnell, K.E., 1997. "Income and the demand for environmental quality," *Environment and development economics,* 2: 383– 399.

Murty, S., 2014. "Necessary and sufficient conditions for an environmental Kuznets curve with some illustrative examples" A draft.

Murty, S., Russell R. R., Levkoff S. B., 2012. "On modeling pollution-generating technologies," *Journal of environmental economics and management*, 64: 117-135.

Panayotou, T., 1993. "Empirical tests and policy analysis of environmental degradation at different stages of economic development," ILO, Technology and Employment Programme, Geneva.

Plassman F., N. Khanna N., 2006. "Preferences, technology, and the environment: Understanding the environmental Kuznets curve hypothesis," *American journal of agricultural economics,* 88(3): 632–643.

Selden, T. M., Song D., 1994. "Environmental quality and development: is there a Kuznets curve for air pollution emissions?" *Journal of environmental economics and management*, 27: 147–162.

Selden, T., Song D., 1995. "Neoclassical growth, the J curve for abatement, and the inverted-U curve for pollution." *Journal of environmental economics and management* 29: 162–168, 1995.

Shafik, N., S. Bandyopadhyay S., 1992. "Economic growth and environmental quality: Time series and cross-country evidence," Background paper for the World Development Report. The World Bank, Washington, DC.

Stokey, N. L., 1998. "Are there limits to growth?" *International economic review*, 39: 1–31.

Stern, D. I., 2004. "The rise and fall of the environmental Kuznets curve," *World development,* 32: 1419–1439.

World Development Report 1992, World Bank, Oxford University Press, New York.