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Abstract

On the properties of an emission-generating technology and its parametric representation.

We propose a set of comprehensive axioms that seek to capture our intuitive understanding of the properties of an emission-generating technology (EGT). We show that an EGT that satisfies these axioms can be parametrically represented by more than one implicit production function that are derived from it. Here, these production functions take the form of distance functions first introduced by Shephard [1953] and Malmquist [1953]. One of these production relations has properties of a neo-classical production function that shows how standard inputs are transformed into standard (intended) outputs. The remaining reflect trade-offs, observed in nature, between emissions, emission-causing goods, and cleaning-up activities of the producing unit. We illustrate this by considering two cases: (i) where each type of cleaning-up activity jointly mitigates all types of emissions and (ii) where cleaning-up activities are emission-specific allowing also for the possibility that a cleaning-up activity while helping to reduce certain emissions, can also contribute to more of some other types of emissions.

Keywords: theory of production, emission-generating technologies, free input and output disposability, weak disposability, costly disposability, functional representations of multi-output production technologies.
JEL classification codes: D20, D24, Q50.

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1. Introduction.

In neo-classical economics, the production technology is a set of all technologically feasible combinations of inputs and outputs. From a given technology set one can derive a production function.[1] If the technology set satisfies a set of axioms, in particular, free-disposability of inputs and outputs, then the production function offers a convenient parametric representation of this set.[2] A violation of these axioms could mean that the production function can no longer be employed to represent the technology set.[3] These required disposability conditions turn out not to be restrictive and they form a part of a set of intuitively appealing axioms that the technology set can be expected to satisfy.[4] In particular, these disposability properties imply that, along the frontier of the technology, which is given by the graph of the function that represents it, the trade-off (technical rate of substitution) between any input and output is non-negative and between any two inputs and between any two outputs is non-positive.

This paper is concerned with the study and identification of analogous properties of the technology set when the production process also generates by-products such as emissions, in addition to standard (which we will also refer to as "intended") outputs. We will call such a technology an emission-generating technology (EGT). In particular, we would like to know what intuitively appealing properties of an EGT ensure that it has a parametric representation, and what is the form that such a representation takes.

This is not the first paper that addresses these questions. The literature has long understood that emissions, which are unintended outputs (or by-products) of the production process, do not satisfy standard output free-disposability. Another characteristic feature of an EGT that seems to have motivated most models of technology in the literature is the

[1] *E.g.,* in a single output and a single input case, from the technology set $Y := \{\langle x, y\rangle \in \mathbf{R}^2_+ \mid x \text{ can produce } y\}$, a production function with image $y = f(x)$, specifying the maximum amount y of the output that can be produced for every level x of the input, can be derived.

[2] In our example, under these axioms, $Y = \overset{*}{Y} := \{\langle x, y\rangle \in \mathbf{R}^2_+ \mid y \leq f(x)\}$.

[3] *E.g.,* in the context of our example, Y may not be equal to $\overset{*}{Y}$, if Y violates free-disposability of the output.

[4] See, *e.g.,* Debreu [1959] or Mas-Colell et al [1995] for a list of such axioms.

intuitive positive trade-off between the production of the intended outputs and emissions: the more are the intended outputs produced, the more are the emissions generated. Classic works such as Shephard [1953] and Baumol and Oates [1988] propose alternative disposability properties for an EGT that can capture these features. Baumol and Oates [1988] and Cropper and Oates [1992] treat emissions as ordinary inputs of production that satisfy free input-disposability. This "input" approach to emission generation has been adopted in several theoretical and applied works.[5] Shephard [1953, 1974] and Färe [1988] model emission as an output whose disposal is not free. Rather, it satisfies (i) *weak-disposability*: reduction of unintended outputs can only be achieved by simultaneous and proportionate reductions of the intended outputs (*i.e.,* technology permits reductions of intended and unintended outputs in a proportionate (radial) manner) and (ii) *null-jointness* (no emission generation implies no production of intended outputs). There is also a huge applied literature that adopts this "output" approach to emission generation.[6]

In both these approaches, all standard inputs and intended outputs satisfy standard input and output free-disposability, respectively. The respective disposability properties adopted by the input and output approaches ensure that the resulting EGTs have parametric representations. In each approach, an EGT can be represented by an implicit or an explicit production function such that, along the graph of the function, the trade-offs between emissions and intended outputs are positive[7] and trade-offs between all other goods have standard (neo-classical) signs. In particular, Shephard [1970, 1974] and Färe [1988] define an "output distance function," which is based on radial expansions of intended and unintended outputs for given levels of inputs, and recommend an implicit functional representation of an EGT satisfying weak-disposability and null-jointness based on this concept.[8]

[5] See, for instance, Reinhard et al. [1999, 2000], Lee et al. [2002], and Hailu and Veeman [2001].

[6] See, *e.g.,* Färe, Grosskopf, Noh, and Yaisawarng [1993], Coggins and Swinton [1994], Murty and Kumar [2002, 2003], Färe, Grosskopf, Lovell, and Pasurka [1989], Färe, Grosskopf, Noh, and Weber [2005], and Boyd and McClelland [1999]. See Zhou and Poh [2008] for a comprehensive survey of a number of papers employing this approach.

[7] *E.g.,* in standard production theory, input free-disposability and output free-disposability imply a positive relationship between an input and an output along the production frontier. Hence, a positive trade-off between emission and intended output will be true in the input approach to emission generation.

[8] Also called a gauge function or a transformation function in the literature, this concept was first introduced independently by Malmquist [1953] and Shephard [1953], and should not be confused with the concept of a metric in mathematics. Distance functions have been extensively studied in the works of

It is intuitive that, in nature, there are some goods that cause emissions, *e.g.,* green house gasses are produced due to the combustion of fossil fuels in intended production. The above two approaches do not distinguish between goods that cause emissions in nature and goods that do not. Murty, Russell, and Levkoff (MRL) [2012] show that an EGT violates free input-disposability of emission-causing inputs, such as fossil fuels, that are used in the production process. Intuitively, using such inputs more than the strictly needed amounts is no longer free, as this also implies generation of greater amounts of emissions. Førsund [2009] and MRL also argue that the use of a single production relation to model an EGT can lead to counter-intuitive trade-offs between various goods in production.[9] MRL further argue that the intuitive positive relation between emission generation and intended output production is, in fact, only a correlation that is effected by the use of emission-causing inputs in intended production: the more are the emission-causing inputs used, the more are emissions generated and the more also is the production of intended outputs. Given this correlation, they argue that an EGT is best represented parametrically by two production relations: one capturing the standard neo-classical transformation of standard inputs into intended outputs and the other capturing the link in nature between emissions and inputs that cause emissions. The overall EGT is, hence, an intersection of a standard neo-classical technology and a nature's emission generating technology.[10]

In this paper, we formally present a comprehensive set of axioms that we feel capture our intuitive understanding of an EGT that results from the simultaneous transformation of inputs into intended outputs and of emission-causing goods into emissions in nature. The emission-causing goods could either be inputs of intended production (such as fossil fuels) or outputs of intended production (such as varieties of cheese that release strong odour as an emission.) Inputs are used not only to produce the intended outputs but also to produce emission-mitigation (cleaning-up) services. Prominent among these axioms are those that characterise the disposability properties of goods that affect both intended production and emission generation in nature, namely the emission-causing goods and

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Gorman [1970], Diewert [1974], Blackorby, Russell, and Primont [1978], and Färe and Primont [1995]. For an excellent discussion and a survey of the concept and its applications, see Russell [1998].

[9] *E.g.,* treating emission as an input implies that the trade-off between emission and an emission-causing input is negative.

[10] The idea of representing some types of production technologies parametrically by more than one production function is not new and goes back to Frisch [1965].

cleaning-up activities. Our axioms seek to capture the fact that, ceteris paribus, changes in the levels of these goods are permitted by an EGT if and only if they are permitted by both the intended production technology and nature's emission-generating mechanism that underlie it. For example, ceteris paribus, if nature's emission-generation permits a change in emission-causing goods, then the overall EGT permits such a change conditional on the intended production technology also permitting such a change.

We show that, if an EGT satisfies these axioms, then it can be represented parametrically by more than one implicit production functions that are derived from it. Each of these functions is defined in the spirit of the distance function introduced by Shephard [1953] and Malmquist [1953]. One of these is a typical neo-classical production function capturing standard production relations between all inputs and outputs. The remaining imply such trade-offs between goods, which our intuition would attribute to nature's emission generation. The overall EGT is an intersection of the intended production technology and the nature's emission-generating set that are defined by these implicit production (distance) functions. In nature, diverse relations can exist between emissions, emission-causing goods, and cleaning-up activities. One or more production relations may be required to capture the diverse trade-offs between such goods. In this paper, we illustrate this by considering two cases – (i) where each type of cleaning-up activity jointly mitigates all types of emissions and (ii) where cleaning-up activities are emission-specific allowing also for the possibility that a cleaning-up activity while helping to reduce certain emissions, can also contribute to more of some other types of emissions.[11] While nature's emission generation in case (i) can be represented by a single distance function, the complex trade-offs between goods in case (ii) imply that more than one distance function are required to represent nature's emission generation set.[12]

The main focus of this paper is on studying the special properties of the technologies that *generate* emissions as byproducts. Starrett [1972] demonstrated that, if emissions from

[11] I am grateful to an anonymous for directing me to study case (ii).

[12] Employing the weak-disposability based output approach to modelling an EGT, Färe, Grosskopf, and Hernandez-Sancho [2004] also derive two distance functions, which are based on radial expansions of the intended output vector holding all other goods fixed vector and radial contractions of the emission vector holding all other goods fixed, to construct an index of environmental performance. However, since the output approach treats both emission-causing and non-emission causing inputs symmetrically *i.e.* assumes that free input disposability holds for all inputs, these two distance functions will imply identical trade-offs between various goods and, hence, will be functionally equivalent.

an emission-generating firm impose detrimental external effects on the intended outputs of another firm (the *emission bearing* firm or the victim), then the technology of the victim firm can also be expected to have non-standard properties; precisely, he demonstrated that this technology exhibits non-convexities. In the context of this paper, we will also consider cases where a production unit's actions to promote its intentions could help or hinder its *own* intentions, in addition to generating potential external effects on the rest of the society, *i.e.,* we will allow for cases when the emissions generated by a production unit can have potential negative or positive effects on its own intended production. In particular, a firm could be *both* a generator and a victim of its own emissions.[13] We show that an EGT exhibits a positive correlation between intended production and emission-generation if the emissions that it generates create either no or positive effects on its own intended production. However, this positive correlation can be absent if its emissions create detrimental effects for its own intended production. In addition, in this case, the EGT will also exhibit Starrett-type non-convexities.

In Section 2, we present a model of an EGT which distinguishes between emission causing and non-emission causing goods and cleaning-up activities. Sections 3 to 6 focus on the case where cleaning-up activities help in jointly mitigating all emissions. In Section 3, we present axioms for an EGT that aim to distinguish between the characteristic features of emission-causing and non-emission causing goods and cleaning-up activities. New disposability conditions need to be introduced to capture the unique features of emission-causing goods and cleaning-up activities. In Section 4, we show the implications of our axioms for the correlation between intended production and emission-generation. In Section 5, we derive two distance functions from an EGT and show that an EGT that satisfies our axioms can be represented parametrically by these two distance functions. In most applied works, it is often convenient to begin with a parametric specification of the technology. To this end, in Section 6, we present some simple examples of pairs of distance functions that can fully specify EGTs that satisfy our axioms. In Section 7, we modify

[13] *E.g.,* a detrimental effect arises when smoke from a steel mill can affect its intended output production by affecting the productivity of its labour. Leguminous plants such as beans and peas are well-known for attracting nitrogen-fixing bacteria which enrich the soil with nitrogen, an important fertilizer in agriculture. This increases the yield of all crops. Thus, nitrogen emissions resulting from cultivation of leguminous crops impose beneficial effects on intended production.

axioms developed in Section 3 to allow for emission-specific cleaning-up activities, derive a functional representation of an EGT for this case, and provide an illustrative example. Some concluding remarks on the implications of our model of EGT are offered in Section 8. Proofs of all results are relegated to the appendix.

2. A model of an EGT.

A starting point of our analysis is a clear distinction between goods that cause emissions in nature and those that do not. Goods that cause emissions in nature could be either inputs of intended production of firms (*e.g.,* fossil fuels such as coal) or outputs of intended production (*e.g.,* varieties of cheese that release strong odour.) In addition, a firm could also engage in activities that clean up its emissions (*e.g.,* scrubbing). These activities are also resource intensive (require inputs). Moslener and Requate [2007] draw attention to different types of cleaning-up activities: Some cleaning-up activities jointly reduce different types of emissions, *e.g.,* catalytic converters in cars jointly reduce SO_2, CO, and NO_x emissions. In some cases, cleaning-up activities could be quite emission-specific, *e.g.,* while SO_2 emissions may be removed by scrubbing, NO_x can be reduced by combustion controls such as low-NO_x burners. Moreover, Moslener and Requate [2007] and Føround [2009] also draw attention to cases where a cleaning-up activity, while reducing one type of emission, can simultaneously increase generation of another type of emission. To capture these features, our model of an EGT has the following components:

(*i*) m intended outputs, of which those indexed $1, \ldots, m_z$ (with $m_z \leq m$) cause emissions and the remaining $m_o = m - m_z$ do not. A quantity vector of intended outputs is denoted by $y = \langle y_z, y_o \rangle \in \mathbf{R}^m_+$. Intended outputs are indexed by j, *e.g.,* y_{z_j} is the quantity of the j^{th} emission-causing intended output.

(*ii*) n inputs, of which those indexed $1, \ldots, n_z$ (with $n_z \leq n$) cause emissions and the remaining $n_o = n - n_z$ do not. A quantity vector of inputs is denoted by $x = \langle x_z, x_o \rangle \in \mathbf{R}^n_+$. Inputs are indexed by i, *e.g.,* x_{z_i} is the quantity of the i^{th} emission-causing input.

(*iii*) m' types of emissions. A quantity vector of emissions is denoted by $z \in \mathbf{R}^{m'}_+$. Emissions are indexed by k, *e.g.,* z_k is the quantity of the k^{th} emission.

(iv) s types of cleaning-up outputs. A quantity vector of cleaning-up outputs is denoted by $a \in \mathbf{R}^s_+$. Abatement outputs are indexed by l, *e.g.,* a_l is the quantity of the l^{th} cleaning-up output.[14]

All inputs are used to produce intended outputs of production or to provide emission-mitigation or cleaning-up services. The use of emission-causing inputs or the production of emission-causing outputs in intended production results in *gross* emission generation in nature, part of which could be mitigated by the cleaning-up activities that the firm undertakes. z denotes emission levels, net of mitigation. An EGT is a set of production vectors of the form $\langle x, a, y, z\rangle = \langle x_z, x_o, a, y_z, y_o, z\rangle \in \mathbf{R}^{n+s+m+m'}_+$ and is denoted by $\mathcal{T} \subset \mathbf{R}^{n+s+m+m'}_+$. Restrictions of $\mathcal{T}$ to various subspaces are denoted by $\mathcal{P}(\cdot)$, *e.g.,* $\mathcal{P}(x, a, z)$ is the restriction of $\mathcal{T}$ to the space of all intended outputs given a fixed quantity vector $\langle x, a, z\rangle$ of all other goods:

$$\mathcal{P}(x, a, z) = \{y \in \mathbf{R}^m_+ \mid \langle x, a, y, z\rangle \in \mathcal{T}\}.$$

Similarly, $\mathcal{P}(x, a, y) \subseteq \mathbf{R}^{m'}_+$ is the restriction of $\mathcal{T}$ to the space of all emissions given a fixed quantity vector $\langle x, a, y\rangle$ of all other goods and $\mathcal{P}(x, a) \subseteq \mathbf{R}^{m+m'}_+$ is the restriction of $\mathcal{T}$ to the space of intended outputs and emissions given a fixed quantity vector $\langle x, a\rangle$ of inputs and cleaning-up activities.

Given fixed amounts of all inputs (which include emission-causing inputs) and cleaning-up activities, we will assume that there are bounds on both the levels of intended outputs and emissions that are technologically feasible. In particular, it is reasonable to assume that there are upper bounds on intended production and lower bounds on emission generation, *e.g.,* a given amount of coal can be associated with a certain maximal amount of electricity and a certain minimal amount of smoke. The latter reflects the fact that it is *costly* to reduce emissions below such a lower bound – further reductions in emissions are not possible if the levels of the emission causing inputs and cleaning-up level are held fixed.

[14] This model also includes the case studied in Rødseth [2013], where some intended outputs themselves help in recuperation of some or all amount of emissions generated by emission causing goods. Thus, these intended outputs also form a part of the set of cleaning-up activities, *i.e.,* their quantities are elements of both vectors $y \in \mathbf{R}^m_+$ and $a \in \mathbf{R}^s_+$.

Technical inefficiency may imply that less intended outputs are produced and more emissions are generated than the levels indicated by the tightest upper and lower bounds on intended outputs and emissions, respectively.[15] One can also envisage upper bounds on emission generation, but if emissions impose detrimental external effects on (other) agents in the economy, then economic efficiency implies that we are more interested in the lower bounds on emission generation.

Definition: $\langle y, z\rangle \in \mathcal{P}(x,a)$ is a strictly efficient point of $\mathcal{P}(x,a)$ if there exists no point $\langle \bar{y}, \bar{z}\rangle \in \mathcal{P}(x,a)$ such that $\langle y, z\rangle \neq \langle \bar{y}, \bar{z}\rangle$, $\bar{y} \geq y$, and $\bar{z} \leq z$.[16] The strictly efficient frontier of $\mathcal{P}(x,a)$, denoted by $Front(\mathcal{P}(x,a))$, is the set of all strictly efficient points of $\mathcal{P}(x,a)$.[17] $\langle x, y, a, z\rangle \in \mathcal{T}$ is a strictly efficient point of $\mathcal{T}$ if $\langle y, z\rangle$ is a strictly efficient point of $\mathcal{P}(x,a)$. The strictly efficient frontier of $\mathcal{T}$ is the set of all strictly efficient points of $\mathcal{T}$.

We define the *costly-disposal hull* of set $\mathcal{T}$ with respect to emissions as

$$T \equiv \{\langle x, y, a, z\rangle \in \mathbf{R}_+^{n+s+m+m'} \mid \exists\, z' \in \mathbf{R}_+^{m'} \text{ such that } z' \leq z \text{ and } \langle x, y, a, z'\rangle \in \mathcal{T}\}.$$

Intuitively, T includes all production vectors in $\mathcal{T}$ as well as production vectors that generate arbitrarily higher levels of emissions than those permitted by $\mathcal{T}$. Restrictions of T to various subspaces are denoted by $P(\cdot)$ and are defined analogously to $\mathcal{P}(\cdot)$, the restrictions of $\mathcal{T}$, defined above. Given the definition of set T, it can be verified that $\mathcal{P}(x,a)$ and $P(x,a)$ have the same set of strictly efficient points.

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15 *E.g.,* the amount of smoke generated by a unit of gas in a car depends on how well the engine is tuned and the manner in which the driver drives the car. Technical inefficiencies in cleaning-up attributed to the the damage done to the cooling system of the reactors by a tsunami, caused more than the minimal levels of radiations to be generated by the nuclear wastes in the Fukushima Daiichi nuclear disaster of 2011. For a more precise engineering and physical-sciences justification of lower bounds on emission generation and environmental inefficiencies, see Baumgärtner and Arons [2003], which defines and discusses thermodynamic inefficiencies.

16 Vector notation: for any two vectors $a = \langle a_1, \ldots, a_n\rangle$ and $b = \langle b_1, \ldots, b_n\rangle$ in an arbitrary Euclidean space $\mathbf{R}^n$,

$$\begin{aligned} a \geq b &\iff a_i \geq b_i \ \forall\ i = 1, \ldots, n, \\ a > b &\iff a_i \geq b_i \ \forall\ i = 1, \ldots, n \text{ with } a \neq b, \text{ and} \\ a \gg b &\iff a_i > b_i \ \forall\ i = 1, \ldots, n. \end{aligned}$$

Where obvious, a vector of zeros will be denoted by 0. However, where the explicit specification of dimensionality becomes important, a vector of zeros of dimension c will be denoted by $0^{(c)}$.

17 Thus, $\langle \bar{y}, \bar{z}\rangle \in \mathcal{P}(x,a)$ is a strictly efficient point of $\mathcal{P}(x,a)$ if there exists no other point in $\mathcal{P}(x,a)$ with no smaller amounts of intended outputs and no bigger amounts emissions.

Remark 1: $Front(\mathcal{P}(x,a)) = Front(P(x,a))$ for all $\langle x,a\rangle \in \mathbf{R}_+^{n+s}$.[18]

Remark 1 implies that we can employ the costly disposal T in place of set $\mathcal{T}$ to study how the lower bounds of emissions change with change in emission-causing inputs and cleaning-up activities. The difference between sets $\mathcal{T}$ and T is that, while the former set does not rule out upper bounds on emission-generation, its costly disposal hull T does. We find that it is easier to derive the intuitive monotonic relations observed in nature between lower bounds on emissions and emission-causing goods if we employ the construct T rather than the actual EGT $\mathcal{T}$.[19] In what follows, we will employ set $\mathcal{T}$ to derive a particular parametric representation of the upper bounds on intended production (function D_1 in Section 5), and we will employ set T to derive particular parametric representation(s) of the lower bounds on emission generation (function D_2 in Section 5 and functions E_1 and E_2 in Section 7). It will turn out that, taken together, the two sets of parametric functions will exactly represent the original EGT $\mathcal{T}$.

Changes in the levels of inputs and cleaning-up activities affect both intended production and emission generation. It will be helpful to identify the set of inputs and cleaning-up levels for which the set of feasible intended output and emission levels are non-empty.

$$\mathbf{\Omega} \equiv \left\{ \langle x,a\rangle \in \mathbf{R}_+^{n+s} \;\middle|\; \mathcal{P}(x,a) \neq \emptyset \right\}.$$

A movement from one combination of inputs and cleaning-up activity in $\mathbf{\Omega}$ to another can change the set of technologically feasible combinations of intended outputs and emissions. For all $\langle x,a\rangle \in \mathbf{\Omega}$, we define the following sets, which we will employ in our analysis below:

$$\begin{aligned}
\mathcal{Y}(x,a) &:= \{y \in \mathbf{R}_+^m \mid \langle y,z\rangle \in \mathcal{P}(x,a) \text{ for some } z \in \mathbf{R}_+^{m'}\},\\
\mathcal{Z}(x,a) &:= \{z \in \mathbf{R}_+^{m'} \mid \langle y,z\rangle \in \mathcal{P}(x,a) \text{ for some } y \in \mathbf{R}_+^{m}\},\\
Z(x,a) &:= \{z \in \mathbf{R}_+^{m'} \mid \langle y,z\rangle \in P(x,a) \text{ for some } y \in \mathbf{R}_+^{m}\}, \text{ and}\\
W(x,a) &:= \left(\mathcal{Z}(x,a) - \mathbf{R}_+^{m'}\right) \cap \mathbf{R}_+^{m'}.
\end{aligned}$$

$\mathcal{Y}(x,a)$ and $\mathcal{Z}(x,a)$ are the projections of $\mathcal{P}(x,a)$ into the space of intended outputs and emissions, respectively, while $Z(x,a)$ is the projections of $P(x,a)$ into the space of

[18] $Front(P(x,a))$ and strictly efficient points of set $P(x,a)$ are defined analogously to the frontier and strictly efficient points of set $\mathcal{P}(x,a)$.

[19] See Theorem (EG) in Section 5.3, where employing set T, they are derived in a manner exactly analogous to the derivation of a positive monotonic relation between an input and an output under the assumptions of input and output free-disposability in a standard neoclassical technology.

emissions. $W(x,a)$ is the free-disposal hull (restricted to the non-negative orthant) of $\mathcal{Z}(x,a)$. Hence, the upper boundary of $\mathcal{Z}(x,a)$ will be a subset of $W(x,a)$. Given the above definitions and notation, Remark 2 follows in an obvious way:

Remark 2: Construction of T from $\mathcal{T}$ implies that

(i) $\mathcal{P}(x,a) \subseteq P(x,a)$ for all $\langle x,a\rangle \in \mathbf{\Omega}$,

(ii) $\mathbf{\Omega} = \left\{ \langle x,a\rangle \in \mathbf{R}_+^{n+s} \;\middle|\; P(x,a) \neq \emptyset \right\}$,

(iii) for all $\langle x,a,y\rangle \in \mathbf{\Omega} \times \mathbf{R}_+^m$, we have $P(x,a,y) = P(x,a,y) + \mathbf{R}_+^{m'} = \mathcal{P}(x,a,y) + \mathbf{R}_+^{m'}$.

(iv) $\mathcal{Y}(x,a) = \{y \in \mathbf{R}_+^m \mid \langle y,z\rangle \in P(x,a) \text{ for some } z \in \mathbf{R}_+^{m'}\}$ for all $\langle x,a\rangle \in \mathbf{\Omega}$, and

(v) $Z(x,a) = \mathcal{Z}(x,a) + \mathbf{R}_+^{m'}$ for all $\langle x,a\rangle \in \mathbf{\Omega}$.

Panel (a) of Figures 1 and 2 explain some of the constructs defined above. Both figures assume there is only one intended output and one type of emission. Holding the levels of all inputs and the cleaning-up activities fixed at $\langle x,a\rangle$, these two sets of figures provide two different examples of the restrictions $\mathcal{P}(x,a)$ of $\mathcal{T}$ and $P(x,a)$ of T. Given the vector $\langle x,a\rangle$, y' is the maximum intended output and z' is the minimum level of emission that can be produced. There is also an upper bound z'' on the amount of emission that can be generated given $\langle x,a\rangle$. $\mathcal{P}(x,a)$ is the bounded area $z'ABz''$. On the other hand, the set $P(x,a)$, which is derived as the costly-disposal hull of the set $\mathcal{P}(x,a)$, has an unbounded area. The line segment $z'A$ is the set of strictly efficient points in $\mathcal{P}(x,a)$ and $P(x,a)$ in Panel (a) of Figure 2. In Panel (a) of Figure 1, A is the only strictly efficient point in $\mathcal{P}(x,a)$ and $P(x,a)$. The set $\mathcal{Y}(x,a)$ is the interval $[0,y']$. The set $\mathcal{Z}(x,a)$ is the bounded interval $[z',z'']$. $Z(x,a)$, the set of emission levels permitted by $P(x,a)$, is the unbounded interval $[z',\infty)$. The set $W(x,a)$ is the interval $[0,z'']$.

3. Some basic axioms: The case of joint reduction of emissions by cleaning-up.

Trade-offs observed between inputs and intended outputs in intended production are distinct from trade-offs observed between emissions and emission-causing goods in nature. In this section we provide a set of axioms for an EGT that results from the simultaneous transformation of inputs into intended outputs and emission-causing inputs and intended outputs into emissions. In this and Sections 4 to 6 we will focus on the case of joint emission reduction by cleaning-up activities.[20] One could expect that these axioms hold for data sets generated from operations of emission-generating production units.

Assumptions (C), (BOUNDy), and (SD-JE$_{x_z,y_z}$) place some standard restrictions on the EGT. In particular, Assumption (BOUNDy) implies that the quantities of intended outputs that can be produced are bounded when the levels of all inputs and the cleaning-up activity are held fixed. Assumption (SD-JE$_{x_z,y_z}$) permits *shutting down (SD)* of the production unit as a technological feasibile option and also captures the joint-essentiality of all emission-causing goods (JE$_{x_z,y_z}$) in producing emissions, *i.e.,* no emission is produced if no emission-causing input is used *and* no emission-causing output is produced.[21] Technologies in Figures 1 and 2 satisfy Assumptions (C), (BOUNDy), and (SD-JE$_{x_z,y_z}$). In particular, Panels (a) in both figures assume $x_z \neq 0$. Panel (b) of Figure 1 assumes that the sole intended output is non-emission generating and that $x_z = 0$, while $x_o \neq 0$. As a result, Assumption (SD-JE$_{x_z,y_z}$) implies that no emission is produced, while there is an upper bound on the production of the intended output. Panel (b) of Figure 2 assumes $x_z = 0$, $x_o \neq 0$, and that the intended output it emission generating. Thus, while the minimal level of emission that can be produced is zero, ceteris paribus, the efficient level of emission increases with increase in the level of the intended output.

Assumption (C): The set $\mathbf{\Omega}$ is closed and convex. The mapping $\boldsymbol{\mathcal{P}} : \mathbf{\Omega} \mapsto \mathbf{R}_+^{m+m'}$ with image $\boldsymbol{\mathcal{P}}(x,a) = \mathcal{P}(x,a)$ is non-empty and continuous.

Assumption (BOUNDy): For all $\langle x, a \rangle \in \mathbf{\Omega}$, the set $\mathcal{Y}(x,a)$ is bounded.

Assumption (SD-JE$_{x_z,y_z}$): $\langle x, a, y, z \rangle \in \mathcal{T}$ and $\langle x_z, y_z \rangle = 0 \implies z = 0$. $\mathcal{T}$ permits shut down, *i.e.,* $0 \in \mathcal{T}$.

We now study the plausible disposability properties of an EGT with respect to various types of goods, *i.e.,* we study what types of changes in the levels of usage or production of these goods, starting from an initially feasible production vector, are permitted by an EGT.

[20] In Section 7, we will modify this set of possible axioms to allow for the case of emission-specific cleaning-up activities.

[21] Compare this with the original null-jointness assumption in Shephard [1953, 1974], Färe [1988], and Färe et al [1988, 1989, 1993], which states that emission generation is zero only if there is no production of intended outputs. When we take account of cleaning-up activities, it is clear that zero emissions are compatible with production of positive amounts of intended outputs.

Two sets of production relations interact to define an EGT: (i) the production relations designed by the production engineer that describe how inputs are transformed into intended outputs and (ii) the laws defined by nature which transform emission-causing goods into emissions. Hence, changes in the levels of different goods are permitted by the overall EGT if and only if they are permitted by *both* (i) and (ii). In this regards, we will argue below that production relations in (i) and the costly-disposal hull of the production relations in (ii) can be expected to satisfy polar opposite disposability properties with respect to some goods, namely the cleaning-up outputs and emission-causing inputs. We will also argue that the overall EGT can be expected to satisfy standard input and output free-disposability with respect to non-emission causing inputs and intended outputs, respectively, and its costly-disposal hull can be expected to satisfy the same with respect to non-emission causing inputs and *all* intended outputs, respectively.

First, we note that nature's emission-causing mechanism is independent of non-emission causing inputs and intended outputs. Hence, ceteris-paribus, technologically feasible changes in the levels of intended outputs or inputs that are non-emission generating have no effect on the lower bounds of emission generation. The axiom below captures this distinguishing property of non-emission causing goods.

Assumption (INDo):

$$\langle x, a\rangle \in \mathbf{\Omega},\ \bar{x}_o \neq x_o,\ \text{and}\ \bar{y}_o \neq y_o \implies P(x,a,y) = P(x_z, \bar{x}_o, a, y_z, \bar{y}_o)$$
$$\text{if } P(x,a,y) \neq \emptyset \text{ and } P(x_z, \bar{x}_o, a, y_z, \bar{y}_o) \neq \emptyset.$$

Figure 1 provides an example of a case where the intended output is non-emission generating. Holding inputs and cleaning-up levels fixed, increasing the intended output from 0 to y' has no effect on the strictly efficient level (lower bound) of emission, which is z'.[22]

Next, the production relations defined in (i) can be assumed to be neo-classical in nature, and hence can be expected to satisfy standard free-disposability with respect to all inputs and outputs, including non-emission causing inputs and intended outputs. Since nature's emission-generating mechanism, which is described by the production relations in

[22] Compare this figure with Figure 2, where the intended output is emission-generating. Here, the lower bound on emission generation increases from z' to $\overset{*}{z}$ as the intended output quantity increases from 0 to y' for fixed levels of all inputs and cleaning-up activity.

(ii), is independent of non-emission causing inputs and intended outputs, we can expect the overall EGT satisfies standard free-disposability with respect to these goods. The axiom below states this.

Assumption (FDo):

$$\langle x_z, x_o, a, y_z, y_o, z\rangle \in \mathcal{T},\ \bar{x}_o \geq x_o,\ \text{and } \bar{y}_o \leq y_o \implies \langle x_z, \bar{x}_o, a, y_z, \bar{y}_o, z\rangle \in \mathcal{T}.$$

In addition, we can also argue that the costly-disposal hull of the overall EGT can be expected to satisfy standard output free-disposability with respect to emission-causing intended outputs, which non-trivially form a part of both production relations (i) and (ii). This is because, ceteris-paribus, arbitrary decreases in these outputs can be explained in terms of technological inefficiencies in both intended production and the costly-disposal hull of nature's emission-generating mechanism. Figure 3 (which abstracts from cleaning-up and assumes only one input and one type of emission) illustrates this: if y'_z is produced by x' and y'_z results in z' levels of emissions, then it is plausible that with technical inefficiencies in *both* intended production and nature's emission-generating mechanism, a lower level of intended output, say y''_z, is both producible with x' under the intended production technology and can continue to produce z' under the costly-disposal hull of nature's emission-generating technology. The axiom below captures this.

Assumption (FDy_z):

$$\langle x_z, x_o, a, y_z, y_o, z\rangle \in T \text{ and } \bar{y}_z \leq y_z \implies \langle x_z, x_o, a, \bar{y}_z, y_o, z\rangle \in T.$$

In the case of emission-causing inputs and cleaning-up outputs that enter both sets of production relations defined by (i) and (the costly-disposal hull of) (ii), it can be expected that while certain changes in the levels of such goods are permitted by one set of production relations, they could be denied by the other. Changes in the levels of different goods are permitted by the overall EGT if and only if they are permitted by both (i) and (ii). Thus, if certain changes in the levels of such goods are permitted by (i) (respectively, (ii)), then these changes are permitted by the overall EGT if and only if they are permitted (*i.e., conditional* only on them being permitted) by (ii) (respectively, (i)).

We first show that the overall EGT can be expected to satisfy only a conditional free-disposability property with respect to emission-causing inputs and cleaning-up outputs.

Since the intended production relations defined in (i) can be assumed to have standard neo-classical properties, the intended production technology can be assumed to satisfy standard input and output free-disposability with respect to emission-causing inputs and cleaning-up outputs, respectively, *i.e.,* ceteris paribus (*i.e.,* holding levels of all other goods fixed), the intended production technology supports arbitrary increases in the levels of emission-causing inputs and arbitrary decreases in the cleaning-up levels.

However, from the point of view of nature's emission-generating mechanism, such increases in inputs and decreases in cleaning-up levels *could be* costly in terms of emission, *i.e.,* they could happen only at the cost of generating more emissions. This will be true, in particular, if the initial production vector $\langle x, a, y, z\rangle \in \mathcal{T}$ is such that z is a lower bound on emission generation given $\langle x, a\rangle$. Intuitively this is because, in that case, combusting more fossil fuels or doing less cleaning-up would mean that the lower bounds on emissions in nature will increase. Hence, if $\bar{x}_z \geq x_z$ and $\bar{a} < a$ then, while y is still feasible in intended production given $\langle \bar{x}_z, x_o, \bar{a}\rangle$ (as intended production satisfies input and output free-disposability), z is no longer feasible under nature's emission-generating mechanism with $\langle \bar{x}_z, x_o, \bar{a}\rangle$. Hence, from the point of view of the overall EGT, $\mathcal{P}(\bar{x}_z, x_o, \bar{a}, z) = \emptyset$. However, if there was some environmental inefficiency in the use of fossil fuels or cleaning-up, so that z was above the lower bounds on emission generation associated with $\langle x, a\rangle$ and was also on or above the lower bounds on emissions associated with $\langle \bar{x}_z, x_o, \bar{a}\rangle$, then nature's emission-generating mechanism could continue generating z amounts of emissions even with the increase in fossil-fuel usage or the decreases in the levels of cleaning-up activities to $\langle \bar{x}_z, \bar{a}\rangle$. Hence, in this case, the overall EGT permits the change in inputs and cleaning-up levels to $\langle \bar{x}_z, x_o, \bar{a}\rangle$ as both intended production and nature's emission-causing mechanism permit it. Hence, $y \in \mathcal{P}(x, a, z)$ implies $y \in \mathcal{P}(\bar{x}_z, x_o, \bar{a}, z)$ and $\mathcal{P}(x, a, z) \subset \mathcal{P}(\bar{x}_z, x_o, \bar{a}, z)$. Thus, conditional on nature's emission-generating mechanism permitting such changes, the set of intended outputs producible under the overall EGT $\mathcal{T}$ expands with increases in the levels of emission-causing inputs and decreases in the levels of cleaning-up. Figure 4 illustrates this when $m = m' = n = s = 1$. The figure assumes that the cleaning-up activity level is held fixed at a and that the sole input is emission generating. An increase in the input level from x to $\bar{x}$ expands the the production possibility set of intended production from A to B, so that the combination $\langle a, y\rangle$ is still feasible in intended production with the new input level $\bar{x}$. However, this may not be true of the emission level in nature. In particular, z (respectively, $\bar{z}$) is the minimal level of emission that x (respectively, $\bar{x}$) level of the emission-causing input can generate. As seen in the figure, $\bar{z} > z$. Hence, $P(\bar{x}, a, z) = \emptyset$. On the other hand, $\bar{z}$ can also be produced by x amount of the input due to environmental inefficiency in using the input. In that case, the figure shows that $y \in P(x, a, \bar{z})$ implies $y \in P(\bar{x}, a, \bar{z})$ and $P(x, a, \bar{z}) \subset P(\bar{x}, a, \bar{z})$.

Assumption (CFD1x_z, a), below, is a *conditional* free-disposability assumption on emission-causing inputs and cleaning-up activity that summarises the arguments above. Assumption (CFD2x_z, a) is imposed on the costly-disposal hull T, and is analogous to Assumption (CFD1x_z, a).

Assumption (CFD1x_z, a):

$$\text{If } \langle x_z, x_o, a, y_z, y_o, z\rangle \in \mathcal{T},\ \bar{x}_z \geq x_z,\ \bar{a} \leq a, \text{ and } \langle \bar{x}_z, \bar{a}\rangle \neq \langle x_z, a\rangle \text{ then}$$
$$\mathcal{P}(\bar{x}_z, x_o, \bar{a}, z) \neq \emptyset \iff \mathcal{P}(x_z, x_o, a, z) \subset \mathcal{P}(\bar{x}_z, x_o, \bar{a}, z).$$

Assumption (CFD2x_z, a):

$$\text{If } \langle x_z, x_o, a, y_z, y_o, z\rangle \in T,\ \bar{x}_z \geq x_z,\ \bar{a} \leq a, \text{ and } \langle \bar{x}_z, \bar{a}\rangle \neq \langle x_z, a\rangle \text{ then}$$
$$P(\bar{x}_z, x_o, \bar{a}, z) \neq \emptyset \iff P(x_z, x_o, a, z) \subset P(\bar{x}_z, x_o, \bar{a}, z).$$

Under Assumption (CFD1x_z, a), ceteris-paribus, an EGT does not permit arbitrary increases in the levels of the emission-generating inputs or arbitrary decreases in the cleaning-up level. Ceteris-paribus, does $\mathcal{T}$, or at least its costly-disposal hull T, also impose plausible restrictions on arbitrary *decreases* in the levels of emission-generating inputs and arbitrary *increases* in the level of cleaning-up?

To answer this, first note that, from the point of view of the costly-disposal hull of nature's emission-generating mechanism, conditions that are polar opposite of standard input and output free-disposability seem plausible for emission-causing inputs and cleaning-up activities, respectively. In Murty [2010] and MRL [2012], these are called *costly disposability of emission-causing inputs and cleaning-up activity*: Suppose $\langle x, a\rangle$ produces $\langle y, z\rangle$. Then, ceteris paribus, levels of emission-causing inputs lower than x and levels of cleaning-up higher than a will be associated, in nature, with lower lower-bounds on emissions. Hence, holding all other inputs fixed, vector z of emissions is also permitted by the costly-disposal hull of nature's emission-generating mechanism when amounts of emission-causing inputs smaller than x and levels of cleaning-up higher than a are used. See left panel of Figure 5 (which is drawn under the same assumptions as Figure 4), where z level of emission can be produced by x amount of fossil fuel and also by a smaller amount $\bar{x}$ of fossil fuel if the fuel is burnt inefficiently. Thus, the costly disposal hull of nature's emission-generating mechanism can be expected to permit arbitrary decreases in emission-causing inputs and arbitrary increases in the levels of the cleaning-up activities.

However, now, intended production may or may not permit such changes. This is because, ceteris paribus, in intended production, y levels of intended outputs may no longer be feasible with the lower levels of fossil fuels or higher level of cleaning-up; *i.e.,* reductions in inputs or increases in the levels of cleaning-up activities *may be* possible in intended production only at the cost of lower levels of intended outputs. Figure 5 shows that the production possibility set of intended production contracts from A to B due to reduction in the input from x to $\bar{x}$. Suppose $y = \hat{y}$. From the point of view of the overall EGT, $z \in P(x, a, \hat{y})$. Although, from the point of view of the costly disposal hull of nature's emission-generating technology, the reduced level of input $\bar{x}$ can generate z, from the point of view of the intended production technology, $\langle \hat{y}, a \rangle$ is an efficient point of A that is not feasible in B. Hence, from the point of view of the overall EGT, we have $P(\bar{x}, a, \hat{y}) = \emptyset$. Now suppose $y = \bar{y}$ in Figure 5. From the point of view of the overall EGT, the figure shows that $z \in P(x, a, \bar{y})$. In this case, $\langle \bar{y}, a \rangle$ is an inefficient point of A, which is also producible in B. At the same time, the costly disposal hull of nature's emission-generating technology permits z to be generated by $\bar{x}$. Hence, from the point of view of the overall EGT, $P(\bar{x}, a, \bar{y}) \neq \emptyset$ and $z \in P(\bar{x}, a, \bar{y})$. Thus, conditional on intended production technology permitting decreases in emission-causing inputs and increases in the levels of the cleaning-up activities, the set of emission levels feasible under the costly disposal hull T expands: in Figure 5, $P(x, a, \bar{y}) \subset P(\bar{x}, a, \bar{y})$. Assumption (CCD$x_z, a$), below, is a conditional costly disposability assumption on the costly disposal hull T of EGT $\mathcal{T}$ that summarises the arguments above.

Assumption (CCDx_z, a):[23]

If $\langle x_z, x_o, a, y, z\rangle \in T,\ \bar{x}_z \leq x_z,\ \bar{a} \geq a,\ \langle \bar{x}_z, \bar{a}\rangle \neq \langle x_z, a\rangle,$ and both x_z and $\bar{x}_z$ are not equal to zero when $m_z = 0$ then $P(\bar{x}_z, x_o, \bar{a}, y) \neq \emptyset \iff P(x_z, x_o, a, y) \subset P(\bar{x}_z, x_o, \bar{a}, y)$.

4. The phenomenon of by-production: The case of joint cleaning-up of emissions.

Much of the literature views a valid EGT as a technology that exhibits a positive relationship between intended production and emission generation. The input and the output approaches discussed in the introductory section propose conditions that result in such a relationship.

An understanding of the role of emission-causing goods in nature, however, shows that this positive relation is, in fact, a *correlation*: the more are the emission-causing inputs used or the less is the cleaning-up, the more are the emissions generated and the more are also the intended outputs produced. A combination of some of our axioms above demonstrates this in Theorem (BP) below. Under the conditions of Theorem (BP), as the levels of emission-causing inputs increase or the levels of the cleaning-up activities decrease, both the sets of feasible levels of intended outputs and feasible levels of emissions shift and the upper bounds on intended outputs and the lower bounds on emissions increase. In Figure 6, it is assumed that all inputs cause the emission. If the input levels increase from x to $\bar{x}$ or cleaning-up levels decrease from a to $\bar{a}$, the upper (respectively, lower) bound on the intended output (respectively, emission) increases from y' to $\bar{y}$ (respectively, from z' to $\bar{z}$). Thus, the set $\mathcal{Y}(x, a) = [0, y']$ expands to the set $\mathcal{Y}(\bar{x}, \bar{a}) = [0, \bar{y}]$ and the set $Z(x, a) = [z', \infty)$ shrinks to the set $Z(\bar{x}, \bar{a}) = [\bar{z}, \infty)$.

Theorem By-production (BP): *Suppose $\langle x_z, x_o, a\rangle \in \mathbf{\Omega}$, $\langle \bar{x}_z, x_o, \bar{a}\rangle \in \mathbf{\Omega}$, $\bar{x}_z \geq x_z$, $\bar{a} \leq a$, and $\langle x_z, a\rangle \neq \langle \bar{x}_z, \bar{a}\rangle$.*

(i) If Assumption (CFD2x_z, a) holds then $\mathcal{Y}(x_z, x_o, a) \subseteq \mathcal{Y}(\bar{x}_z, x_o, \bar{a})$.

(ii) If Assumptions (FDo), (FDy_z), (SD-JE$_{x_z,y_z}$), and (CCDx_z, a) hold then $Z(\bar{x}_z, x_o, \bar{a}) \subseteq Z(x_z, x_o, a)$.

[23] If Assumption (SD-JE$_{x_z,y_z}$) also holds and $x_z = \bar{x}_z = 0$ when $m_z = 0$, then $P(x_z, x_o, a, y) = P(\bar{x}_z, x_o, \bar{a}, y) = \{0^{m'}\}$. Assumption (CCD$x_z, a$) excludes this case.

It is possible that the emissions generated by a firm when it undertakes intended production (and triggers off nature's emission-generating mechanism) can, in turn, affect its own production of intended outputs. Assumption (INDz) captures the case where such effects are absent: the set of intended outputs that are feasible under technology $\mathcal{T}$ for given levels of inputs, cleaning-up activity, and emissions is unaffected by changes in its emissions. The technology in Figure 1 satisfies (INDz).

Assumption (INDz):

$$\langle x, a\rangle \in \mathbf{\Omega} \text{ and } \bar{z} \neq z \implies \mathcal{P}(x,a,z) = \mathcal{P}(x,a,\bar{z}) \text{ if } \mathcal{P}(x,a,z) \neq \emptyset \text{ and } \mathcal{P}(x,a,\bar{z}) \neq \emptyset.$$

Assumptions (DETz) and (BENz) capture the case where emissions generated by a firm impose detrimental or beneficial effects of on its own intended production, respectively: the set of intended outputs that are feasible under technology $\mathcal{T}$ for given levels of inputs, cleaning-up activity, and emissions shrinks or expands, respectively, as its emissions increase. The technology in Figure 7 satisfies (DETz), while that in Figure 8 satisfies (BENz).[24]

Assumption (DETz):

$$\langle x, a\rangle \in \mathbf{\Omega},\ \bar{z} \geq z, \text{ and } \bar{z} \neq z \implies \mathcal{P}(x,a,\bar{z}) \subset \mathcal{P}(x,a,z) \text{ if } \mathcal{P}(x,a,z) \neq \emptyset \text{ and } \mathcal{P}(x,a,\bar{z}) \neq \emptyset.$$

Assumption (BENz):

$$\langle x, a\rangle \in \mathbf{\Omega},\ \bar{z} \geq z, \text{ and } \bar{z} \neq z \implies \mathcal{P}(x,a,z) \subset \mathcal{P}(x,a,\bar{z}) \text{ if } \mathcal{P}(x,a,z) \neq \emptyset \text{ and } \mathcal{P}(x,a,\bar{z}) \neq \emptyset.$$

Starrett [1972] demonstrated that if one firm (the generator) creates detrimental external effects on the production of intended outputs of some other firm (the victim), then the victim's technology exhibits non-convexities. In our present context, Assumption (DETz) points to a case where a firm can be both the generator and a victim of its own actions, namely, its generation of emissions that negatively affect its own intended production. In Starrett's case, the victim's technology is defined over infinity of potential levels of external effects the victim may face, and is, hence, unbounded in the direction of

[24] If a plot of empirically observed data from an EGT leads to Figure 2, then one can draw either of two conclusions: (i) Assumption (BENz) holds or (ii) the intended output is emission generating. Hence, based only on the empirically observed data, these two cases are indistinguishable. Prior engineering and scientific knowledge is required to make such distinctions.

the detrimental external effect. In our case, where the victim firm is also the generator, it is intuitive that, for fixed amounts of emission-causing goods used by the firm, there will be a bounded range of emission levels. Figure 7, which assumes $m' = m = 1$, $n = n_z = 1$, and $s = 0$ shows possible structures of the set $\mathcal{P}(x)$ when Assumption (DETz) holds. Panel (a) (respectively, (b)) shows the case where $\mathcal{P}(x)$ is non-convex (respectively, convex). These structures are explained in Example 2 in Section 6.

Lemma 1 and Corollary to Theorem (BP), below, show that the positive correlation between emissions and intended outputs exists when emissions impose no or positive effects on intended production. However, it *may not* exist when emissions impose detrimental effects on intended production.

Lemma 1: *If Assumption (INDz) or Assumption (BENz) hold then* $\mathcal{P}(x, a, z) = P(x, a, z)$. *If, in addition, Assumption (CFD1*x_z, a*) holds then Assumption (CFD2*x_z, a*) is also true.*

Corollary of Theorem (BP): *Suppose* $\mathcal{T}$ *satisfies Assumptions (CFD1*x_z, a*), (FDo), (FD*y_z*), (SD-JE*$_{x_z,y_z}$*), and (CCD*x_z, a*). If, in addition, either Assumption (INDz) or Assumption (BENz) hold, then for all* $\langle x_z, x_o, a\rangle \in \mathbf{\Omega}$ *and* $\langle \bar{x}_z, x_o, \bar{a}\rangle \in \mathbf{\Omega}$ *such that* $\bar{x}_z \geq x_z$, $\bar{a} \leq a$, *and* $\langle x_z, x_o, a\rangle \neq \langle \bar{x}_z, x_o, \bar{a}\rangle$, *we have*

$$\mathcal{Y}(x_z, x_o, a) \subseteq \mathcal{Y}(\bar{x}_z, x_o, \bar{a}) \text{ and } Z(\bar{x}_z, x_o, \bar{a}) \subseteq Z(x_z, x_o, a).$$

In the case when emissions cause detrimental external effects, Figures 9 and 10 demonstrate that the positive correlation may not or may exist, respectively. It is assumed that all inputs are emission generating, $\bar{x} > x$, and $\bar{a} \leq a$. In both cases, Assumption (CFD1x_z, a) is true, *e.g.,* $\mathcal{P}(x, a, \bar{z}) \subset \mathcal{P}(\bar{x}, \bar{a}, \bar{z})$. However, in Figure 9, conclusions of Lemma 1 fail: $P(x, a, \bar{z}) \neq \mathcal{P}(x, a, \bar{z})$. Rather, $P(x, a, \bar{z}) = \mathcal{P}(x, a, z')$. But $P(\bar{x}, \bar{a}, \bar{z}) = \mathcal{P}(\bar{x}, \bar{a}, \bar{z})$. Hence, Assumption (CFD2x_z, a) is violated: $P(x, a, \bar{z}) \not\subset P(\bar{x}, \bar{a}, \bar{z})$.

5. Distance-function representations of EGT: The case of joint cleaning-up of emissions.

Employing our assumptions above, here we present a functional representation of an EGT that is based on the concept of a distance function. Since an EGT reflects both intended production by firms and the nature's emission-generating mechanism, two distance functions D_1 and D_2 are derived from it.

D_1 will capture the (upper) bounds set by intended production on intended outputs and hence will be defined relative to $\mathcal{T}$ and its restrictions $\mathcal{P}(\cdot)$. Define the mapping

$$D_1 : \mathbf{\Omega} \times \mathbf{R}_+^{m+m'} \longmapsto \mathbf{R}_+ \cup \ \{\infty\} \tag{5.1}$$

with image

$$D_1(x, a, y_z, y_o, z) \equiv \inf \ \{\lambda_1 > 0 \bigm| \langle \frac{y}{\lambda_1}, \frac{z}{\lambda_1} \rangle \in \mathcal{P}(x, a)\}.$$

D_2 will capture the (lower) bounds set by nature on emission generation and hence will be defined relative to the costly-disposal hull T and its restrictions $P(\cdot)$. Define the mapping

$$D_2 : \mathbf{\Omega} \times \mathbf{R}_+^{m+m'} \longmapsto \mathbf{R}_+ \cup \ \{\infty\} \tag{5.2}$$

with image

$$D_2(x, a, y_z, y_o, z) := \inf \ \{\lambda_2 \geq 0 \bigm| \langle \lambda_2 y, \lambda_2 z \rangle \in P(x, a)\}.$$

Define the mappings: $\Lambda_1 : \mathbf{\Omega} \times \mathbf{R}_+^{m+m'} \mapsto \mathbf{R}_+$ and $\Lambda_2 : \mathbf{\Omega} \times \mathbf{R}_+^{m+m'} \mapsto \mathbf{R}_+$ whose images are the constraint sets of the optimization problems (5.1) and (5.2):

$$\begin{aligned}\Lambda_1(x, a, y, z) :=& \{\lambda_1 > 0 \bigm| \langle \frac{y}{\lambda_1}, \frac{z}{\lambda_1} \rangle \in \mathcal{P}(x, a)\} \text{ and} \\ \Lambda_2(x, a, y, z) :=& \{\lambda_2 > 0 \bigm| \langle \lambda_2 y, \lambda_2 z \rangle \in P(x, a)\}.\end{aligned}$$

The domains where the images of these mappings are non-empty are defined as follows:

$$\begin{aligned}\mathbf{\Lambda_1} &= \{\langle x, a, y, z \rangle \in \mathbf{\Omega} \times \mathbf{R}_+^{m+m'} | \ \Lambda_1(x, a, y, z) \neq \emptyset\} \text{ and} \\ \mathbf{\Lambda_2} &= \{\langle x, a, y, z \rangle \in \mathbf{\Omega} \times \mathbf{R}_+^{m+m'} | \ \Lambda_2(x, a, y, z) \neq \emptyset\}.\end{aligned}$$

5.1. Three additional axioms.

Assumptions (UBz), (R1), and (R2) are additional conditions that will be employed in the theorems below. (UBz) characterizes plausible changes in the upper bounds on emission generation due to increases in the levels of emission-causing inputs or a decrease in the level of cleaning-up activity. It postulates that, in these situations, the upper bounds on emissions increase.[25]

Assumption (UBz):

$$\langle x_z, x_o, a\rangle \in \mathbf{\Omega},\ \langle \bar{x}_z, x_o, \bar{a}\rangle \in \mathbf{\Omega},\ \bar{x}_z \geq x_z,\ \bar{a} \leq a,\ \text{and}\ \langle \bar{x}_z, \bar{a}\rangle \neq \langle x_z, a\rangle \implies W(x,a) \subset W(\bar{x}, \bar{a}).$$

Assumption (UBz) is illustrated in Figure 11, where $W(x,a) = [0, z''] \subset [0, \tilde{z}] = W(\bar{x}, \bar{a})$.

Assumption (R1):

$$\langle x, a\rangle \in \mathbf{\Omega} \text{ and } z \in W(x,a) \setminus \mathcal{Z}(x,a) \implies z' \notin \mathcal{Z}(x,a)\ \forall\ z' < z.$$

Define the set

$$\mathbf{\Theta} := \{\langle x, a, y\rangle \in \mathbf{\Omega} \times \mathbf{R}^m \ \big|\ P(x,a,y) \neq \emptyset\}. \tag{5.3}$$

Assumption (R2):

$$\langle x, a, y\rangle \in \mathbf{\Theta} \text{ and } z \in P(x,a,y) \setminus \mathcal{P}(x,a,y) \implies z' \notin \mathcal{P}(x,a,y)\ \forall\ z' > z.$$

If there is only one type of emission ($m' = 1$) and the EGT was convex then Assumptions (R1) and (R2) hold trivially. Violations of these assumptions for $m' = 2$, even when the EGT is convex, are illustrated in Figures 12 and 13, respectively. In Figure 12, $b \in W(x,a) \setminus \mathcal{Z}(x,a)$ and there exist points smaller than b in $\mathcal{Z}(x,a)$. In Figure 13, $b \in P(x,a,y) \setminus \mathcal{P}(x,a,y)$ and there exist points bigger than b in $\mathcal{P}(x,a,y)$.

Figures 14 to 16 (a) illustrate EGTs where Assumptions (R1) and (R2) are satisfied.[26] Together, Assumptions (R1) and (R2) imply that the trade-off between any two types of

[25] Unlike in the case of the lower bounds, this property of the upper bounds of emission generation is stated directly as an axiom rather than derived. This is because, in this paper we are not focussing on the study of the upper bounds on emission generation.

[26] Note, under free disposability of intended outputs (*i.e.,* if Assumptions (FDo) and (FDy_z) hold), it can be shown that Assumption (R2) implies (R2'): $z \in Z(x,a) \setminus \mathcal{Z}(x,a) \implies z' \notin \mathcal{Z}(x,a)\ \forall\ z' > z$. Figures 14 to 16(a) assume (FDo) and (FDy_z) and demonstrate (R1) and (R2') implied by (R2).

emissions along the lower boundary of emission generation, when all inputs and cleaning-up levels are held fixed, is non-positive, *i.e.,* one emission is not the cause of another emission.[27] Two emissions can be substitutes for one another as in Figures 14 and 15[28] or can be independent of each other as in Figure 16.

5.2. Properties of mapping D_1 and extracting the intended production technology underlying an EGT.

For any production vector $\langle x, a, y, z\rangle \in \mathbf{R}_+^{n+s+m+m'}$, D_1 asks the question, what is the least amount by which we need to radially scale down the output vector $\langle y, z\rangle$ to make the resulting vector producible under technology $\mathcal{T}$ with input level x and cleaning-up level a. Thus, $\langle \frac{y}{D_1(x,a,y,z)}, \frac{z}{D_1(x,a,y,z)}\rangle$ is in the set $\mathcal{P}(x, a)$. In Figure 17, this scaling process takes points such as $\langle z, y\rangle$, $\langle z', y'\rangle$, and $\langle \bar{z}, \bar{y}\rangle$ to $\langle \overset{*}{z}, \overset{*}{y}\rangle \in \mathcal{P}(x, a)$. Thus, the minimal scaling-down factors are $D_1(x, a, y, z) = \frac{\|\langle z,y\rangle\|}{\|\langle \overset{*}{z},\overset{*}{y}\rangle\|} < 1$, $D_1(x, a, y', z') = \frac{\|\langle z',y'\rangle\|}{\|\langle \overset{*}{z},\overset{*}{y}\rangle\|} < 1$, and $D_1(x, a, \bar{y}, \bar{z}) = \frac{\|\langle \bar{z},\bar{y}\rangle\|}{\|\langle \overset{*}{z},\overset{*}{a}\rangle\|} > 1$, respectively.[29] In each of these cases, the constraint set of optimization (5.1) is non-empty (*e.g.,* $\Lambda_1(x, a, y, z)$ is the set of scaling factors which move $\langle z, y\rangle$ along the ray r), and D_1 is well defined and unique. However, the constraint set of problem (5.1) can also be empty; *e.g.,* $\Lambda_1(x, a, \tilde{y}, \tilde{z}) = \emptyset$ and by definition of D_1 in (5.1), $D_1(x, a, \tilde{y}, \tilde{z}) = \infty$. Given an EGT, the values taken by distance function D_1 can be conditioned by any existing upper bounds on emission generation in nature. In Figure 17, the upper bound on emission in the set $\mathcal{P}(x, a)$ is z''. The function D_1 scales any emission-intended output combination that lies on or below ray r'', to a point on the line-segment point Az'' where emission-level is equal to the upper bound z'' (*e.g.,* it scales down $\langle \hat{z}, \hat{y}\rangle$ to $\langle z'', y''\rangle$).

Theorem (IP) states the properties of the mapping D_1 that is derived from an EGT satisfying some of our axioms above by employing problem (5.1).[30]

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[27] This is not true in segments AB of $\mathcal{Z}(x, a)$ in Figures 12 and 13.

[28] *E.g.,* depending upon the amount of oxygen levels available, the more is the amount of carbon monoxide produced by burning a given amount of coal, the less is the amount of carbon-dioxide produced.

[29] $\| \langle z, y\rangle \|$ is the Euclidean norm of the vector $\langle z, y\rangle$.

[30] Proofs of some properties of functions D_1 and D_2 are similar to the ones found in the literature for standard distance functions. These are relegated to the working paper version Murty [2012]. Proofs of the rest of the properties follow as a consequence of our special axioms.

Theorem Intended production (IP): *Suppose Assumptions (C), (BOUNDy), (FDo), (CFD1x_z, a), (FDy_z), (UBz), and (R1) hold. Then the following are true:*

(i) $D_1(x,a,y,z)$ *is well defined (finite) and unique if and only if* $\langle x,a,y,z\rangle \in \mathbf{\Lambda_1}$.

(ii) $\langle x,a,y,z\rangle \in \mathcal{T} \Rightarrow D_1(x,a,y,z) \leq 1$.

(iii) Restricted to the domain $\mathbf{\Lambda_1}$*:*

- D_1 *is homogeneous of degree one. It is convex in* y *and* z *if* $\mathcal{P}(x,a)$ *is convex;*
- D_1 *is non-decreasing in* y*;*
- D_1 *is non-increasing in* x *and non-decreasing in* a
- D_1 *is constant in* z *for all* $\langle x,a,y,z\rangle \in \mathbf{\Lambda_1}$ *such that* $\frac{z}{D_1(x,a,y,z)} \in Int(\mathcal{Z}(x,a))$ *if Assumption (INDz) holds*[31] *;*
- D_1 *is non-decreasing in* z *if Assumption (DETz) holds; and*
- D_1 *is jointly continuous in its arguments if Assumption (DETz) or (INDz) is true.*

We now employ D_1 to extract the intended production technology $\mathcal{T}_1$ underlying an EGT $\mathcal{T}$. Define the set

$$\mathcal{T}_1 = \{\langle x,a,y,z\rangle \in \mathbf{\Omega} \times \mathbf{R}_+^{m+m'} \mid D_1(x,a,y,z) \leq 1\}. \tag{5.4}$$

Under the assumptions of Theorem (IP) above, the set $\mathcal{T}_1$ embodies all production relations that are typical of an intended production technology. To see this assume D_1 is differentiable in the interior of $\mathbf{\Lambda_1}$. Then, from Theorem (IP), evaluated at any production vector $\langle x,a,y,z\rangle \in \mathbf{\Omega} \times \mathbf{R}_+^{m+m'}$ such that $D_1(x,a,y,z) = 1$, the trade-off between any input i and any intended output j is non-negative ($-\frac{\partial D_1/\partial x_i}{\partial D_1/\partial y_j} \geq 0$). Similarly, that between any intended output and cleaning-up is non-positive and that between any two inputs is non-positive.[32] The trade-off between any intended output and emission is zero or negative depending on whether $\mathcal{T}$ satisfies Assumptions (INDz) or (DETz), respectively. Hence, the implicit function $D_1(x,a,y,z) = 1$ is a typical neo-classical production function that represents the set $\mathcal{T}_1$. Hence, $\mathcal{T}_1$ is a typical neoclassical intended production technology.

[31] Given any subset A of a Euclidean space $\mathbf{R}^k$, $IntA$ is the interior of A relative to $\mathbf{R}^k$.

[32] I am thankful to an anonymous referee for drawing attention to the case when intended outputs are unaffected by changes in the levels of cleaning-up activities, *i.e.*, $-\frac{\partial D_1/\partial a_l}{\partial D_1/\partial y_j} = 0$ could be possible. Intuitively, as discussed in Førsund [2009], this is true if inputs that are used for cleaning-up activities are distinct from inputs that produce the intended outputs. In that case the inputs released due to reductions in cleaning-up levels cannot be employed to increase production of intended outputs. There are also empirical works such as Shadbegian and Gray [2005] that provide evidence that increases in cleaning-up activities need not adversely affect intended output production.

5.3. Properties of mapping D_2 and extracting nature's emission generation set underlying an EGT.

For any production vector $\langle x, a, y, z\rangle \in \mathbf{R}_+^{n+s+m+m'}$, D_2 asks the question, what is the least amount by which we need to radially scale up the output vector $\langle y, z\rangle$ to make it feasible to produce under the costly-disposal hull T of technology $\mathcal{T}$ with input level x and cleaning-up level a. Thus, $\langle D_2(x,a,y,z)y, D_2(x,a,y,z)z\rangle$ is in the set $P(x,a)$.

In Figure 18, this scaling process takes points such as $\langle z,y\rangle$, $\langle z',y'\rangle$, and $\langle \bar{z},\bar{y}\rangle$ to $\langle \hat{z},\hat{y}\rangle \in \mathcal{P}(x,a)$. Thus, the minimal scaling-up factors are $D_2(x,a,y,z) = \frac{\|\langle \hat{z},\hat{y}\rangle\|}{\|\langle z,y\rangle\|} < 1$, $D_2(x,a,y',z') = \frac{\|\langle \hat{z},\hat{y}\rangle\|}{\|\langle z',y'\rangle\|} > 1$, and $D_2(x,a,\bar{y},\bar{z}) = \frac{\|\langle \hat{z},\hat{y}\rangle\|}{\|\langle \bar{z},\bar{y}\rangle\|} < 1$, respectively. In each of these cases, the constraint set of optimization (5.2) is non-empty (*e.g.,* $\Lambda_2(x,a,y,z)$ is the set of scaling factor which move $\langle z,y\rangle$ along the ray r), and D_2 is unique and well defined (real valued). However, the constraint set of problem (5.2) can be empty; as *e.g.,* $\Lambda_2(x,a,\tilde{y},\tilde{z}) = \emptyset$ and by definition of D_2 in (5.2), $D_2(x,a,\tilde{y},\tilde{z}) = \infty$.

Theorem (EG) states the properties of the mapping D_2 that is derived from an EGT satisfying some of our axioms above by employing problem (5.2).

Theorem Emission generation (EG): *Suppose Assumptions (C), (FDo), (FDy_z), (SD-JE$_{x_z,y_z}$), and (CCDx_z,a) hold. Then the following are true:*

(i) *$D_2(x,a,y,z)$ is well defined (finite) and unique if and only if $\langle x,a,y,z\rangle \in \mathbf{\Lambda_2}$.*

(ii) *$\langle x,a,y,z\rangle \in T \Rightarrow D_2(x,a,y,z) \leq 1$.*

(iii) *Restricted to the domain $\mathbf{\Lambda_2}$:*

- *D_2 is homogeneous of degree one and convex in y and z if $\mathcal{P}(x,a)$ is convex;*
- *D_2 is non-increasing in z and non-decreasing in y;*
- *D_2 is non-decreasing in x and non-increasing in a if Assumption (CFD2x_z,a) holds;*
- *D_2 is constant in y_o and x_o if Assumption (INDo) holds; and*
- *D_2 is jointly continuous in its arguments if $m_z = 0$.*

We employ D_2 to extract the costly disposal hull of nature's emission generation set T_2 underlying an EGT $\mathcal{T}$. Define the set

$$T_2 = \{\langle x,a,y,z\rangle \in \mathbf{\Omega} \times \mathbf{R}_+^{m+m'} \mid D_2(x,a,y,z) \leq 1\}. \tag{5.5}$$

If $D_2(x, a, y, z) = 1$ for $\langle x, a, y, z\rangle \in \mathbf{\Omega} \times \mathbf{R}_+^{m+m'}$, then z denotes the lower bounds on emissions in nature associated with $\langle x, a, y\rangle$. Since D_2 is non-increasing in the levels of emissions, given $\langle x, a, y\rangle$, T_2 permits all emission vectors bigger than z. Under the assumptions of Theorem (EG) above, the set T_2 embodies all production relations seen in nature between emission-causing goods and emissions. To see this, assume D_2 is differentiable in the interior of $\mathbf{\Lambda_2}$. Then, from Theorem (EG), evaluated at any production vector $\langle x, a, y, z\rangle \in \mathbf{\Omega} \times \mathbf{R}_+^{m+m'}$ such that $D_2(x, a, y, z) = 1$, the trade-off between any emission type k and any emission-causing input i is non-negative ($-\frac{\partial D_2/\partial x_i}{\partial D_2/\partial z_k} \geq 0$). Similarly, that between any type of emission and cleaning-up is non-positive, that between any type of emission and any emission-causing intended output is non-negative, and that between any type of emission and any non-emission causing input or intended output is zero. Hence, the implicit function $D_2(x, a, y, z) = 1$ satisfies all properties that one expects of the lower frontier of nature's emission generation set. Hence, T_2 can be considered to be nature's emission generation set.

5.4. Functional representation of an EGT and its efficient frontier.

We now define a by-production technology based on the fundamental disposability axioms for an EGT that have been postulated above and derive its functional representation.

Definition: $\mathcal{T}$ is a *by-production technology (BPT)* if Assumptions (C), (BOUNDy), (FDo), (FDy_z), (SD-JE$_{x_z,y_z}$), (CCDx_z, a), (CFD1x_z, a), (UBz), (INDo), and (R1) hold.

5.4.1. A representation theorem for a BPT.

Theorem (BP-REPR), below, shows that a given BPT $\mathcal{T}$ can be represented functionally by employing both the distance functions defined above. Hence, it can be decomposed into its underlying intended production technology set $\mathcal{T}_1$ and nature's emission generation set T_2.

Theorem Representation (BP-REPR): *Suppose $\mathcal{T}$ is a BPT, Assumption (R2) holds, and the mapping $\boldsymbol{\mathcal{P}} : \boldsymbol{\Omega} \mapsto \mathbf{R}_{+}^{m+m'}$ with image $\boldsymbol{\mathcal{P}}(x,a) = \mathcal{P}(x,a)$ is convex valued. If D_1 and D_2 are defined as in (5.1) and (5.2), respectively, then*

$$\langle x,a,y,z\rangle \in \mathcal{T} \iff D_1(x,a,y,z) \leq 1 \text{ and } D_2(x,a,y,z) \leq 1. \tag{5.6}$$

If $\mathcal{T}_1$ and T_2 are defined as in (5.4) and (5.5), then

$$\mathcal{T} = \mathcal{T}_1 \cap T_2. \tag{5.7}$$

5.4.2. The strictly efficient points of a BPT.

Theorem (BP-EFF), below, provides a characterization of the efficient points of a BPT.

Theorem Efficiency (BP-EFF): *Suppose $m_z = 0$, $\mathcal{T}$ is a BPT, and Assumption (R2) holds. If either Assumption (INDz) or Assumption (DETz) hold and D_1 and D_2 are defined as in (5.1) and (5.2), respectively, then the following is true:*
$\langle y,z\rangle$ is a strictly efficient point of $\mathcal{P}(x,a) \implies D_1(x,a,y,z) = 1$ and $D_2(x,a,y,z) = 1$.

In Figures 1 and 7 point A is the only strictly efficient point in $\mathcal{P}(x,a)$. The conclusions of this theorem do not follow when emissions impose beneficial external effects or when there exist emission-causing outputs. In Figures 2 and 8, the set of strictly efficient points in $\mathcal{P}(x,a)$ is the line-segment $z'A$. At all such points, excluding point A, D_2 takes value equal to 1. However, this is not true of D_1.

6. Some numerical examples: Constructing BPTs.

In the first two examples below, we present EGTs that satisfy the definition of a BPT. We derive the distance functions D_1 and D_2 defined in Problems (5.1) and (5.2).

In the above discussion, we began with a set theoretic characterization of a BPT and then derived it's functional representation by employing programmes (5.1) and (5.2). In most applied works, however, it is convenient to begin with a parametric specification of the technology. We can do so by specifying two real-valued continuous functions D_1 and D_2 with properties listed in Parts (iii) of Theorems (IP) and (EG), respectively. The underlying intended production technology set and the costly-disposal hull of the nature's emission generation set are defined then by (5.4) and (5.5), respectively. The intersection of these two sets will yield a BPT. Examples 3 and 4 adopt this approach.

Example 1. Consider a case where $m_z = 0$, $m_o = 1$, $m' = 1$, $n_z = n_o = 1$, $s = 0$, and the EGT is given by $\mathcal{T} = \left\{ \langle x_z, x_o, y, z \rangle \in \mathbf{R}^4_+ \;\middle|\; y \leq \sqrt{x_z x_o} \quad \text{and} \quad \alpha x_z \leq z \leq \beta x_z \right\}$, where $\beta > \alpha > 0$. Note that this EGT imposes both upper and lower bounds on emission generation. It can be verified that $\mathcal{T}$ is a BPT. In particular, let's verify that it satisfies axioms (CFD1x_z, a) and (CCDx_z, a).

Let $y \in \mathcal{P}(x_z, x_o, z)$. Then $y \leq \sqrt{x_z x_o}$ and $\beta x_z \geq z \geq \alpha x_z$. Suppose $\bar{x}_z > x_z$. Then $y \leq \sqrt{x_z x_o} < \sqrt{\bar{x}_z x_o}$. Also, $z \leq \beta x_z < \beta \bar{x}_z$. Two cases are possible: (i) $z \geq \alpha \bar{x}_z$ and (ii) $z < \alpha \bar{x}_z$. If case (i) is true then $\langle \bar{x}_z, x_o, y, z \rangle \in \mathcal{T}$. Hence, $y \in \mathcal{P}(\bar{x}_z, x_o, z)$ so that $\mathcal{P}(x_z, x_o, z) \subset \mathcal{P}(\bar{x}_z, x_o, z)$. Suppose case (ii) is true. Then there exists no $y' > 0$ such that $\langle \bar{x}_z, x_o, y', z \rangle \in \mathcal{T}$. Hence, $\mathcal{P}(\bar{x}_z, x_o, z) = \emptyset$. This proves that $\mathcal{T}$ satisfies (CFD1x_z, a).

Let $z \in P(x_z, x_o, y)$. Then $y \leq \sqrt{x_z x_o}$ and $z \geq \alpha x_z$. Suppose $\bar{x}_z < x_z$. Then $z \geq \alpha x_z > \alpha \bar{x}_z$. Two cases are possible: (i) $y \leq \sqrt{\bar{x}_z x_o}$ and (ii) $y > \sqrt{\bar{x}_z x_o}$. If case (i) is true then $\langle \bar{x}_z, x_o, y, z \rangle \in T$. Hence, $z \in P(\bar{x}_z, x_o, y)$ so that $P(x_z, x_o, y) \subset P(\bar{x}_z, x_o, y)$. Suppose case (ii) is true. Then there exists no $z' > 0$ such that $\langle \bar{x}_z, x_o, y, z' \rangle \in T$. Hence, $P(\bar{x}_z, x_o, y) = \emptyset$. This proves that $\mathcal{T}$ satisfies (CCDx_z, a).

Functions D_1 and D_2 are derived by employing (5.1) and (5.2), for the case when $\langle x_z, x_o, y, z \rangle \gg 0$ as

$$
\begin{aligned}
D_1(x_z, x_o, y, z) &:= \inf \left\{ \lambda_1 > 0 \;\middle|\; \frac{y}{\lambda_1} \leq \sqrt{x_z x_o},\ \alpha x_z \leq \frac{z}{\lambda_1} \leq \beta x_z \right\} \\
&= \inf \left\{ \lambda_1 > 0 \;\middle|\; \frac{y}{\sqrt{x_z x_o}} \leq \lambda_1,\ \frac{z}{\beta x_z} \leq \lambda_1 \leq \frac{z}{\alpha x_z} \right\}. \\
\implies D_1(x_z, x_o, y, z) &= \frac{y}{\sqrt{x_z x_o}} \quad \text{if} \quad \frac{z}{\beta x_z} \leq \frac{y}{\sqrt{x_z x_o}} \leq \frac{z}{\alpha x_z} \\
&= \frac{z}{\beta x_z} \quad \text{if} \quad \frac{y}{\sqrt{x_z x_o}} < \frac{z}{\beta x_z} \\
&= \infty \quad \text{if} \quad \frac{y}{\sqrt{x_z x_o}} > \frac{z}{\alpha x_z}
\end{aligned}
$$

$$
\begin{aligned}
D_2(x_z, x_o, y, z) &:= \inf \left\{ \lambda_2 \geq 0 \;\middle|\; \lambda_2 y \leq \sqrt{x_z x_o},\ \alpha x_z \leq \lambda_2 z \leq \beta x_z \right\} \\
&= \inf \left\{ \lambda_2 \geq 0 \;\middle|\; \frac{\sqrt{x_z x_o}}{y} \geq \lambda_2,\ \frac{\alpha x_z}{z} \leq \lambda_2 \leq \frac{\beta x_z}{z} \right\}.
\end{aligned}
$$

$$\implies D_2(x_z, x_o, y, z) = \frac{\alpha x_z}{z} \quad \text{if} \quad \frac{\alpha_z x_z}{z} \leq \frac{\sqrt{x_z x_o}}{y}$$
$$= \infty \quad \text{if} \quad \frac{\alpha x_z}{z} > \frac{\sqrt{x_z x_o}}{y}$$

The important point to note is that, given input levels $\langle x_z, x_o \rangle$, distance function D_1 is conditioned not only by the upper bound on intended production, $\sqrt{x_z x_o}$, but also by the upper bound on emission, βx_z. Figure 17 illustrates this: D_1 expands the emission and intended output combination $\langle z^o, y^o \rangle$ to $\langle z'', y'' \rangle$, where z'' is the upper bound on emission and y'' is less than $\overset{*}{y}$, the maximal feasible intended output that can be produced given the inputs. At the same time, D_2 is not altered by the upper bound on emission. It can be verified that distance functions D_1 and D_2 have properties listed in Parts (iii) of Theorems (IP) and (EG), respectively. Employing (5.4) and (5.5), they can be employed to define sets $\mathcal{T}_1$ and T_2.

Alternatively, we could have started with following two distance functions, which have properties listed in Parts (iii) of Theorems (IP) and (EG), respectively.

$$D_1(x_z, x_o, y, z) = \frac{y}{\sqrt{x_z x_o}} \quad \text{and} \quad D_2(x_z, x_o, y, z) = \frac{\alpha x_z}{z}, \quad \alpha > 0, \langle x_z, x_o, z \rangle \in \mathbf{R}^3_{++}.$$

Then (5.4) and (5.5) imply

$$\mathcal{T}_1 := \{\langle x_z, x_o, y, z \rangle \in \mathbf{R}^4_+ \mid y \leq \sqrt{x_z x_o}\} \quad \text{and} \quad T_2 := \{\langle x_z, x_o, y, z \rangle \in \mathbf{R}^4_+ \mid z \geq \alpha x_z\}. \tag{6.1}$$

Example 1′. Consider a modification of Example 1, where the EGT is given by $\mathcal{T} = \left\{\langle x_z, x_o, y, z \rangle \in \mathbf{R}^4_+ \mid y \leq \sqrt{x_z x_o} \quad \text{and} \quad z \geq \alpha x_z\right\}$, where $\alpha > 0$. Note that, in this case, set $\mathcal{T}$ imposes no upper bound on emission-generation, and so it coincides with its costly disposal hull, *i.e.*, $\mathcal{T} = T$.

Functions D_1 and D_2 are derived by employing (5.1) and (5.2), for the case when $\langle x_z, x_o, y, z \rangle \gg 0$ as

$$D_1(x_z, x_o, y, z) := \inf \{\lambda_1 > 0 \mid \frac{y}{\lambda_1} \leq \sqrt{x_z x_o},\ \frac{z}{\lambda_1} \geq \alpha x_z\} = \inf \left\{\lambda_1 > 0 \mid \frac{y}{\sqrt{x_z x_o}} \leq \lambda_1 \leq \frac{z}{\alpha x_z}\right\}$$
$$\implies D_1(x_z, x_o, y, z) = \frac{y}{\sqrt{x_z x_o}} \quad \text{if} \quad \frac{y}{\sqrt{x_z x_o}} \leq \frac{z}{\alpha x_z}$$
$$= \infty \quad \text{if} \quad \frac{y}{\sqrt{x_z x_o}} > \frac{z}{\alpha x_z}$$

$$D_2(x_z, x_o, y, z) := \inf\,\{\lambda_2 \geq 0 \mid \lambda_2 y \leq \sqrt{x_z x_o},\ \lambda_2 z \geq \alpha x_z\} = \inf\left\{\lambda_2 \geq 0 \mid \frac{\sqrt{x_z x_o}}{y} \geq \lambda_2 \geq \frac{\alpha x_z}{z}\right\}$$

$$\begin{aligned} \implies D_2(x_z, x_o, y, z) &= \frac{\alpha x_z}{z} \quad \text{if} \quad \frac{\alpha x_z}{z} \leq \frac{\sqrt{x_z x_o}}{y} \\ &= \infty \quad \text{if} \quad \frac{\alpha x_z}{z} > \frac{\sqrt{x_z x_o}}{y} \end{aligned}$$

The point to note is that D_1 is determined totally by the upper bound on intended output production in this case where there are no upper bounds on emission generation.

Example 2. Through this example, we will illustrate Starrett [1972]-type technological non-convexities when emissions of a firm have detrimental effects on its own intended production. We assume $n_z = 1$, $n_o = 0$, $m_z = m = 1$, $m' = 1$, and $s = 0$. The EGT $\mathcal{T}$ and its costly disposal hull T are given by

$$\begin{aligned} \mathcal{T} &= \left\{\langle x_z, y, z\rangle \in \mathbf{R}^3_+ \mid y \leq \max\{0, \sqrt{x_z} - \gamma z\} \quad \text{and} \quad \alpha x_z \leq z \leq \beta x_z\right\} \\ T &= \left\{\langle x_z, y, z\rangle \in \mathbf{R}^3_+ \mid y \leq \max\{0, \sqrt{x_z} - \gamma z\} \quad \text{and} \quad z \geq \alpha x_z\right\}, \end{aligned}$$

where $0 < \alpha < \beta$.

In the set $\mathcal{T}$, the level of emission is both bounded from above and below for any level of the input. The possible structures of the set $\mathcal{P}(x_z)$ are illustrated in Figures 7 (a) and (b). They show how the input transforms into the intended output and that emission of the firm has a negative effect on the intended output produced by the firm. The level of the intended output of the firm is restricted to be non-negative. In the figures, $\overset{*}{z}$ is such that $\sqrt{x_z} - \gamma \overset{*}{z} = 0$, *i.e.*, $\overset{*}{z} = \sqrt{x_z}/\gamma$. The figures demonstrate that $\mathcal{P}(x_z)$ is non-convex if and only if z^* is less than the upper bound on emission $\beta x_z = z''$ in nature.

It can be verified that $\mathcal{T}$ is a BPT. The distance functions D_1 and D_2 derived from $\mathcal{T}$ by employing (5.1) and (5.2) for $\langle x_z, z\rangle \in \mathbf{R}^2_{++}$ and $y \in \mathbf{R}_+$ are given below for the case $\sqrt{x_z} - \gamma\alpha x_z > 0$ and $\langle x_z, y, z\rangle \gg 0$.

$$D_1(x_z, y, z) = \inf\left\{\lambda_1 > 0 \mid \lambda_1 \geq \frac{y + \gamma z}{\sqrt{x_z}},\ \frac{z}{\beta x_z} \leq \lambda_1 \leq \frac{z}{\alpha x_z}\right\}$$

$$\begin{aligned} \implies D_1(x_z, y, z) &= \frac{z}{\beta x_z} \quad \text{if} \quad \frac{y + \gamma z}{\sqrt{x_z}} \leq \frac{z}{\beta x_z} \\ &= \frac{y + \gamma z}{\sqrt{x_z}} \quad \text{if} \quad \frac{z}{\alpha x_z} < \frac{y + \gamma z}{\sqrt{x_z}} \leq \frac{z}{\beta x_z} \\ &= \infty \quad \text{if} \quad \frac{z}{\alpha x_z} < \frac{y + \gamma z}{\sqrt{x_z}}. \end{aligned}$$

$$
\begin{aligned}
& D_2(x_z, y, z) = \inf\left\{\lambda_2 \geq 0 \;\middle|\; \lambda_2 y \leq \sqrt{x_z} - \gamma\lambda_2 z,\ \alpha x_z \leq \lambda_2 z \leq \beta x_z\right\} \\
\implies & D_2(x_z, y, z) = \inf\left\{\lambda_2 \geq 0 \;\middle|\; \lambda_2 \leq \frac{\sqrt{x_z}}{y + \gamma z},\ \frac{\alpha x_z}{z} \leq \lambda_2 \leq \frac{\beta x_z}{z}\right\} \\
\implies & D_2(x_z, y, z) = \frac{\alpha x_z}{z} \quad \text{if} \quad \frac{\sqrt{x_z}}{y + \gamma z} \geq \frac{\alpha x_z}{z} \\
& \qquad\qquad\qquad = \infty \quad \text{if} \quad \frac{\sqrt{x_z}}{y + \gamma z} < \frac{\alpha x_z}{z}.
\end{aligned}
$$

Starrett [1972] demonstrated that non-convexities in the technology sets of victim firms facing detrimental external effects from other firms implied failure of Arrowian/Coasian markets for rights to emit. Correction of such external effects requires government intervention. However, in the case when a firm is both the generator and the victim of its own emissions, it will internalise emission decisions as its emission generation affects its own intended production.[33] No government intervention is required to regulate the firm to correct for the effects of its own emissions on its own intended production. Government intervention would still be required to correct for the external effects created by emissions generated by other economic agents on this firm.[34]

Example 3. Example 1 can be extended to incorporate a cleaning-up activity of the firm and an emission-generating output by defining the images of functions D_1 and D_2 as $D_1(x_z, x_o, a, y_z, y_o, z) = \frac{y_z + y_o}{\sqrt{x_z x_o} - a}$ and $D_2(x_z, x_o, a, y_z, y_o, z) = \frac{\alpha x_z - \theta a}{z - \delta y_z}$ with $\alpha > 0$ with $\theta > 0$ and $\delta > 0$. If a production vector $\langle x_z, x_o, a, y_z, y_o, z\rangle$ satisfies $D_1(x_z, x_o, a, y_z, y_o, z) = 1$ and $D_2(x_z, x_o, a, y_z, y_o, z) = 1$, then we have $y_z + y_o + a = \sqrt{x_z x_o}$ and $z = \alpha x_z + \delta y_z - \theta a$.

Example 4. In the case of multiple emissions (*i.e.,* $m' > 1$), we consider some plausible trade-offs between different types of emissions in nature for fixed levels of emission-generating goods. Under the assumptions of Theorem (EG), D_2 is non-increasing in every type of emission. Two possible cases are illustrated below. We assume $m' = 2$, $m_z = 0$, $n_z = n = 1$, $s = 0$.

(1) In nature, the trade-off between emissions one and two is negative. *E.g.,* there could be a negative relationship between the amount of carbon-dioxide and carbon-monoxide generated when a ton of coal is burnt – the more is one gas generated, the less is

[33] I am grateful to an anonymous referee for drawing my attention to this point.

[34] Further examples of BPT which include cleaning-up activities and multiple emissions can be found in the working paper version Murty [2012].

the other. Suppose $D_2(x,y,z_1,z_2) = \frac{x_z}{z_1 z_2}$, then the trade-off in nature between the two emissions when the emission-causing input is held fixed is given by $-\frac{\partial D_2/\partial z_1}{\partial D_2/\partial z_2} = -\frac{z_2}{z_1} < 0$ if $z_1 \neq 0$ and $z_2 \neq 0$. Figure 15 illustrates this case.

(2) There could also be complementarities in the production of the two types of emissions. *E.g.,* a given variety of coal may possess certain amounts of sulphur in addition to its carbon content, so that every ton of such a coal when burnt (jointly) releases both carbon dioxide and sulphur dioxide. The following functional form for D_2 illustrates such a relationship for the fixed proportion case:

$$D_2(x,y,z_1,z_2) = \frac{x_z}{\min\{\phi_1 z_1, \phi_2 z_2\}} \leq 1 \iff x_z \leq \min\{\phi_1 z_1, \phi_2 z_2\}.$$

It is as if there were two separate relationships in nature: $z_1 = \phi_1 x_z$ and $z_2 = \phi_2 x_z$, which are simultaneously triggered off when the emission-generating input is employed by the firm. Figure 16 illustrates this case.

7. An EGT with emission-specific cleaning-up (ESCU).

We now consider the case where cleaning-up activities are very specific to the type of the emission and where it is possible that while a cleaning-up activity reduces some type of emission, it increases generation of some other types. To sharpen our focus and for notational ease, without loss of generality, we assume the case of two emissions and two cleaning-up activities, *i.e.,* $m' = 2$ and $s = 2$. The first (respectively, the second) cleaning-up output helps mitigate the first (respectively, the second) emission. At the same time, it is possible that the first (respectively, the second) cleaning-up output can contribute to an increase in the second (respectively, first) emission. For every emission of type $k = 1, 2$, z_{-k} denotes the quantity of emission that is not of type k. In this section, we provide axioms that capture the emission-specific nature of cleaning-up activities and provide a functional representation of an EGT with emission-specific cleaning-up. Proofs of most results in this section closely follow the steps of proofs in Sections 4 and 5. These can be found in the working paper version Murty [2012].

7.1. Modifications of some existing axioms, some new axioms, and the phenomenon of by-production.

We now modify some of our earlier axioms and also state some additional axioms to capture the features of an EGT with emission-specific cleaning-up.

Under Assumption (EG-INDz), below, ceteris paribus, generation of any type of emission is independent of generation of the other type of emission, *i.e.,* holding all other goods (including emission-generating goods) fixed, changes in one type of emission have no effect on the generation of the other type of emission.[35] In Panel (b) of Figure 16, the costly disposal hull of the set of feasible emission levels of type two (respectively, one), holding inputs, cleaning-up output, and intended output levels fixed at $\langle x, a, y \rangle \in \mathbf{\Theta}$, is (z_2', ∞) (respectively, (z_1', ∞)) irrespective of any feasible level of emission of type one, *e.g.,* z_1' and z_1'' (respectively, two, *e.g.,* z_2' and z_2'').[36]

Assumption (EG-INDz): For $k = 1, 2$,

$$\langle x, a, y \rangle \in \mathbf{\Theta} \text{ and } \bar{z}_k \neq z_k \implies P(x, a, y, z_k) = P(x, a, y, \bar{z}_k)$$
$$\text{if } P(x, a, y, z_k) \neq \emptyset \text{ and } P(x, a, y, \bar{z}_k) \neq \emptyset.$$

For all $\langle x, a, y \rangle \in \mathbf{\Theta}$ and for $k = 1, 2$, we define the set of all feasible levels of emission of type k as:

$$Z_k(x, a, y) := \{z_k \in \mathbf{R}_+ \mid \langle z_k, z_{-k} \rangle \in P(x, a, y) \text{ for some } z_{-k} \in \mathbf{R}_+\}.$$

Part (iii) of Remark 2 implies that $Z_k(x, a, y)$ is a closed and convex interval that is unbounded from above and bounded from below for any $k = 1, 2$ and $\langle x, a, y \rangle \in \mathbf{\Theta}$. *E.g.,* in Panel (b) of Figure 16, for $k = 1, 2$, the set $Z_k(x, a, y)$ is (z_k', ∞), which has these properties. This figure also demonstrates Remark 3 below:

Remark 3: Under Assumption (EG-INDz), $P(x, a, y) = Z_1(x, a, y) \times Z_2(x, a, y)$ for all $\langle x, a, y \rangle \in \mathbf{\Theta}$.

[35] *E.g.,* when a variety of coal with sulphur and carbon content is burnt, the amounts of sulphur dioxide that could possibly be generated could be independent of the amounts of carbon-dioxide that could possibly be generated. This assumption makes the analysis below more tractable and notationally simple. However, we conjecture that the analysis can also be extended to the general case where, ceteris paribus, there is also dependence in the generation of different types of emissions.

[36] Recall the definition of $\mathbf{\Theta}$ in (5.3).

In the case when cleaning-up activities jointly reduced emissions, the conditional costly disposal assumption (CCDx_z, a), ceteris paribus, permitted arbitrary decreases in emission-causing inputs and arbitrary increases in cleaning-up outputs, conditional upon these changes being feasible in intended production. However, when each cleaning-up output no longer reduces all emissions jointly, conditional costly disposability of cleaning-up activities will need to be modified. However, conditional costly disposability of emission-causing inputs can still be expected to hold for emission-causing inputs. Hence, we restate this assumption for emission-causing inputs.

Assumption (CCDx_z):

If $\langle x_z, x_o, a, y, z\rangle \in T$ and $\bar{x}_z \leq x_z$ then $P(\bar{x}_z, x_o, a, y) \neq \emptyset \iff P(x_z, x_o, a, y) \subset P(\bar{x}_z, x_o, a, y)$.

Assumption (ESCU) below captures emission-specific nature of cleaning-up activities. Ceteris paribus, an increase in the level of a cleaning-up output (conditional on being feasible in intended production) while decreasing the lower bound of one of the emission, may cause an increase in the lower bound for the other emission-type.

Assumption (ESCU): For $k = 1, 2$,

$$\langle x, a_k, a_{-k}, y\rangle \in \mathbf{\Theta}, \quad \bar{a}_k > a_k, \quad \text{and} \quad \langle x, \bar{a}_k, a_{-k}, y\rangle \in \mathbf{\Theta}$$
$$\implies Z_k(x, a_k, a_{-k}, y) \subset Z_k(x, \bar{a}_k, a_{-k}, y) \text{ and } Z_{-k}(x, \bar{a}_k, a_{-k}, y) \subseteq Z_{-k}(x, a_k, a_{-k}, y).$$

Figure 19 illustrates this assumption for $k = 1$. It assumes $m = 1$ and that the levels of the intended output, inputs, and cleaning-up output of type two are held fixed, say at $\langle x, a_2, y\rangle$. Panel (a) shows that starting from a_1 level of cleaning-up output of type one, ceteris paribus, an increase to $\bar{a}_1$ level is permitted by the intended production technology (due to technical inefficiency in intended production at $\langle a_1, y\rangle$). Panels (b) and (c) show that such an increase decreases the lower bound on emission of type one from z_1 to $\bar{z}_1$ and increases the lower bound on emission of type two from z_2 to $\bar{z}_2$.

In the case with emission-specific cleaning-up, the correlation between intended-output production and emission generation that was summarised in Theorem (BP) will have to be modified. Theorem (BP-ESCU) below is such a modified version of Theorem (BP). Part (i) of Theorem (BP) on expansion of intended output production due to an increase in the levels of emission-causing inputs or a decrease in production of cleaning-up activities continues to hold. Part (ii) of Theorem (BP) on the increases in the lower bounds on emission-generation now holds only with respect to increases in the the levels of emission-causing inputs. Assumptions (EG-INDz) and (ESCU) and Remark 3 imply that, ceteris paribus, as cleaning-up output of type k increases (respectively, decreases), the lower bound on emission of type k decreases (respectively, increases) and that of emission of type $-k$ increases (respectively, decreases). Panel (d) of Figure 19 shows this case when the cleaning-up output of type one increases from a_1 to $\bar{a}_1$.

Theorem By-production (BP-ESCU):

(i) *If* $\langle x_z, x_o, a\rangle \in \mathbf{\Omega}$, $\langle \bar{x}_z, x_o, \bar{a}\rangle \in \mathbf{\Omega}$, $\bar{x}_z \geq x_z$, $\bar{a} \leq a$, $\langle x_z, a\rangle \neq \langle \bar{x}_z, \bar{a}\rangle$, *and Assumption (CFD2*$x_z, a$*) holds then* $\mathcal{Y}(x_z, x_o, a) \subseteq \mathcal{Y}(\bar{x}_z, x_o, \bar{a})$.

(ii) *If* $\langle x_z, x_o, a\rangle \in \mathbf{\Omega}$, $\langle \bar{x}_z, x_o, a\rangle \in \mathbf{\Omega}$, $\bar{x}_z > x_z$, *and Assumptions (FDo) and (CCD*x_z*) hold then* $Z(\bar{x}_z, x_o, a) \subseteq Z(x_z, x_o, a)$.

(iii) *For* $k = 1, 2$, *if* $\langle x_z, x_o, a_k, a_{-k}, y\rangle \in \mathbf{\Theta}$, $\langle x_z, x_o, \bar{a}_k, a_{-k}, y\rangle \in \mathbf{\Theta}$, $\bar{a}_k < a_k$, *and Assumptions (EG-IND*z*) and (ESCU) hold then* $Z_k(x_z, x_o, \bar{a}_k, a_{-k}, y) \subseteq Z_k(x_z, x_o, a_k, a_{-k}, y)$ *and* $Z_{-k}(x_z, x_o, a_k, a_{-k}, y) \subseteq Z_{-k}(x_z, x_o, \bar{a}_k, a_{-k}, y)$.

7.2. Distance function representation of EGT.

To capture the right trade-offs between cleaning-up activities and emissions in the case when cleaning-up is emission specific and when there is also a possibility that increase in cleaning-up of one type can increase emission of the other type, more than one functional relation is required to model the emission-generating mechanism of nature. We show that such an EGT is represented parametrically by three functional relations, when $m' = 2$: The intended-production technology $\mathcal{T}_1$ continues to be represented by distance function D_1 defined in (5.1). For each emission of type $k = 1, 2$, a separate distance function E_k is defined as

$$E_k : \mathbf{\Theta} \times \mathbf{R}_+ \mapsto \mathbf{R}_+ \ \cup \ \{\infty\}$$

with image

$$E_k(x, a, y, z_k) := \inf \ \left\{\gamma_k \geq 0 \ \middle| \ \gamma_k z_k \in Z_k(x, a, y)\right\}. \tag{7.1}$$

For each $k = 1, 2$, define the mapping: $\Gamma_k : \boldsymbol{\Theta} \times \mathbf{R}_+ \mapsto \mathbf{R}_+$ whose image is the constraint set of the optimization problem (7.1):

$$\Gamma_k(x, a, y, z_k) := \{\gamma_k \geq 0 \mid \gamma_k z_k \in Z_k(x, a, y)\}.$$

In Panel (d) of Figure 19, $Z_1(x, a, y) = [z_1', \infty]$. $\Gamma_1(x, a, y, z_1'') \neq \emptyset$ and $E_1(x, a, y, z_1'') = \frac{z_1'}{z_1''}$. The following theorem states the properties of function E_k for $k = 1, 2$.

Theorem Emission generation (EG-ESCU): *Suppose Assumptions (C), (FDo), (FDy$_z$), (INDo), (CCDx$_z$), (EG-INDz), and (ESCU) hold. Then, for any $k = 1, 2$, the following are true:*

(i) *$E_k(x, a, y, z_k)$ is well defined (finite) and unique if and only if $\langle x, a, y\rangle \in \boldsymbol{\Theta}$ and ($z_k > 0$ or $0 \in Z_k(x, a, y)$).*[37]

(ii) $\langle x, a, y, z\rangle \in T \Longrightarrow E_k(x, a, y, z_k) \leq 1$.

(iii) *Restricted to the domain where it is well defined:*

- *E_k is homogeneous of degree one and convex in z_k;*
- *E_k is non-increasing in z_k;*
- *E_k is non-decreasing in y;*
- *E_k is non-decreasing in x_z;*
- *E_k is non-increasing in a_k;*
- *E_k is non-decreasing in a_{-k};*
- *E_k is constant in y_o and x_o; and*
- *E_k is jointly continuous in its arguments.*

We employ functions E_k for $k = 1, 2$ to extract the costly disposal hull of nature's emission generation set T_2 underlying an EGT $\mathcal{T}$. Define the set

$$T_2 = \{\langle x, a, y, z\rangle \in \boldsymbol{\Theta} \times \mathbf{R}_+^2 \mid E_k(x, a, y, z_k) \leq 1 \;\forall\; k = 1, 2\}. \tag{7.2}$$

Thus T_2 is represented now by two functions, each describing the mechanism by which a particular type of emission is generated in nature. If $E_k(x, a, y, z_k) = 1$ for $\langle x, a, y, z_k\rangle \in \boldsymbol{\Theta} \times \mathbf{R}_+$ and for any $k = 1, 2$, then z_k is the lower bound on emission of type k that is associated with $\langle x, a, y\rangle$. Under the assumptions of Theorem (EG-ESCU) above, the

[37] Hence, $E_k(x, a, y, z_k)$ is not well defined if and only if $P(x, a, y) = \emptyset$ or ($z_k = 0$ and $0 \notin Z_k(x, a, y)$).

function E_k embodies all the trade-offs that our intuition suggests exist between emission of type k and other goods when cleaning-up is emission-specific and can also generate emission of the other type. *E.g.*, evaluated at any production vector $\langle x, a, y, z_k \rangle \in \mathbf{\Omega} \times \mathbf{R}_+$ such that $E_k(x, a, y, z_k) = 1$, the trade-offs between emission of type k and any emission-causing input i is non-negative ($-\frac{\partial E_k / \partial x_{z_i}}{\partial E_k / \partial z_k} \geq 0$). The trade-off between emission of type k and cleaning-up activity of type k is non-positive, that between emission of type k and cleaning-up activity of type $(-k)$ is non-negative, and that between emission of type k and any emission-causing intended output is non-negative.

7.3. Functional representation of an EGT with emission-specific cleaning-up and its efficient frontier.

We now define a by-production technology with emission-specific cleaning up based on the modified set of axioms discussed above.

Definition: $\mathcal{T}$ is a *by-production technology with emission-specific cleaning-up (BPT-ESCU)* if Assumptions (C), (BOUNDy), (FDo), (FDy_z), (CCDx_z), (CFD1x_z, a), (EG-INDz), (ESCU), (INDo), (UBz), and (R2) hold.

7.3.1. A representation theorem for a BPT-ESCU.

Theorem (BP-ESCU-REPR), below, shows that a given BPT-ESCU $\mathcal{T}$ can be represented functionally by employing the three distance functions defined above. Hence, it can be decomposed into its underlying intended production technology set $\mathcal{T}_1$ and nature's emission generation set T_2.

Theorem Representation (BP-ESCU-REPR): *Suppose $\mathcal{T}$ is a BPT-ESCU and Assumption (R2) holds. If D_1, E_1, and E_2 are defined as in (5.1) and (7.1), respectively, then*

$$\langle x, a, y, z \rangle \in \mathcal{T} \iff \langle x, a, y, z \rangle \in \mathcal{T}_1 \cap T_2, \tag{7.3}$$

where $\mathcal{T}_1$ and T_2 are defined as in (5.4) and (7.2).

7.3.2. The strictly efficient points of a BPT-ESCU.

Theorem (BP-ESCU-EFF), below, provides a characterization of the efficient points of a BPT-ESCU.

Theorem Efficiency (BP-ESCU-EFF): *Suppose $\mathcal{T}$ is a BPT-ESCU and Assumption (R2) holds. If either Assumption (INDz) or Assumption (DETz) hold and D_1 and E_k for $k = 1, 2$ are defined as in (5.1) and (7.1), respectively, then the following is true:*
$\langle y, z\rangle$ is a strictly efficient point of $\mathcal{P}(x, a) \implies D_1(x, a, y, z) = 1$ and $E_k(x, a, y, z_k) = 1$ for $k = 1, 2$.

7.4. An example:

Example 5. $m_z = 0$, $m_o = 1$, $m' = 2$, $n_z = n_o = 1$, and $s = 2$. It can be verified that the following EGT is a BPT-ESCU.

$$\mathcal{T} = \Big\{\langle x_z, x_o, a_1, a_2, y, z_1, z_2\rangle \in \mathbf{R}^7_+ \ \Big|\ y \le \max\{0, \sqrt{x_z x_o} - a_1 - a_2\},\ z_1 a_1 \ge x_z a_2,$$
$$z_2 \ge \max\{\alpha x_z - \beta a_2 + \mu a_1,\ 0\}\Big\}, \quad \alpha > 0,\ \beta > 0,\ \mu > 0.$$

Problems (5.1) and (7.1) yield the following distance functions for the case when $\sqrt{x_z x_o} - a_1 - a_2 > 0$, $\alpha x_z - \beta a_2 + \mu a_1 > 0$, $x_z > 0$, $a_2 > 0$, $z_2 > 0$, and $z_1 > 0$:

$$D_1(x, a, y, z) = \inf \left\{\lambda_1 > 0 \ \Big|\ \lambda_1 \ge \frac{y}{\sqrt{x_z x_o} - a_1 - a_2},\ \lambda_1 \le \frac{z_1 a_1}{x_z a_2},\ \lambda_1 \le \frac{z_2}{\alpha x_z - \beta a_2 + \mu a_1}\right\}$$

$$\implies D_1(x, a, y, z) = \frac{y}{\sqrt{x_z x_o} - a_1 - a_2} \quad \text{if} \quad \frac{y}{\sqrt{x_z x_o} - a_1 - a_2} \le \min\left\{\frac{z_1 a_1}{x_z a_2},\ \frac{z_2}{\alpha x_z - \beta a_2 + \mu a_1}\right\}$$
$$= \infty \quad \text{if} \quad \frac{y}{\sqrt{x_z x_o} - a_1 - a_2} > \min\left\{\frac{z_1 a_1}{x_z a_2},\ \frac{z_2}{\alpha x_z - \beta a_2 + \mu a_1}\right\}.$$

$$E_1(x, a, y, z_1) = \inf_{\gamma_1, z_2} \left\{\lambda_1 \ge 0 \ \Big|\ y \le \sqrt{x_z x_o} - a_1 - a_2,\ \gamma_1 \ge \frac{x_z a_2}{z_1 a_1},\ z_2 \ge \alpha x_z - \beta a_2 + \mu a_1\right\}$$

$$\implies E_1(x, a, y, z_1) = \frac{x_z a_2}{z_1 a_1} \quad \text{if} \quad y \le \sqrt{x_z x_o} - a_1 - a_2$$
$$= \infty \quad \text{if} \quad y > \sqrt{x_z x_o} - a_1 - a_2.$$

$$E_2(x, a, y, z_2) = \inf_{\gamma_2, z_1} \left\{\lambda_1 \ge 0 \ \Big|\ y \le \sqrt{x_z x_o} - a_1 - a_2,\ z_1 a_1 \ge x_z a_2,\ \gamma_2 \ge \frac{\alpha x_z - \beta a_2 + \mu a_1}{z_2}\right\}$$

$$\implies E_2(x, a, y, z_2) = \frac{\alpha x_z - \beta a_2 + \mu a_1}{z_2} \quad \text{if} \quad y \le \sqrt{x_z x_o} - a_1 - a_2$$
$$= \infty \quad \text{if} \quad y > \sqrt{x_z x_o} - a_1 - a_2.$$

Alternatively, a BPT-ESCU can be constructed by specifying three real-valued continuous functions distance functions D_1, E_1, and E_2 with properties listed in Parts (iii) of Theorems (IP) and (EG-ESCU). The underlying intended production technology set and the costly-disposal hull of the nature's emission generation set are defined then by (5.4) and (7.2), respectively. The intersection of these two sets will yield a BPT-ESCU.

Thus, the following distance functions yield a BPT-ESCU.

$$\begin{aligned}
D_1(x,a,y,z) &= \frac{y}{\sqrt{x_z x_o} - a_1 - a_2}, \\
E_1(x,a,y,z_1) &= \frac{x_z a_2}{z_1 a_1}, \\
E_2(x,a,y,z_2) &= \frac{\max\{\alpha x_z - \beta a_2 + \mu a_1,\ 0\}}{z_2},\ \ \alpha > 0,\ \beta > 0,\ \mu > 0.
\end{aligned}$$

This implies that $\langle x,a,y,z\rangle \in \mathcal{T}$ if and only if

$$y + a_1 + a_2 \leq \sqrt{x_z x_o}, \quad z_1 \geq \frac{x_z a_2}{a_1}, \quad \text{and} \quad z_2 \geq \max\{\alpha x_z - \beta a_2 + \mu a_1,\ 0\}.$$

8. Concluding remarks.

This paper has attempted to highlight the importance of distinguishing between emission-causing goods, non-emission causing goods, and cleaning-up activities while modelling an EGT. While it is intuitive and well-accepted that by-products such as emissions are not freely disposable, what is not commonly realised and established is the fact that emission-causing inputs in nature (such as fossil fuels) and cleaning-up activities will also not satisfy standard input and output free-disposability, respectively. Ceteris paribus, increases in the use of emission-causing inputs or decreases in production of cleaning-up levels come at the cost of greater emissions. Our axioms, which attempt to capture these features, imply that an EGT comprises of two distinct sets of production relations: (i) that captures the relations designed by human-engineers that define how inputs are transformed into intended outputs and (ii) that captures the laws that govern how emission-causing goods are transformed into emissions in nature and the extent to which our cleaning-up activities mitigate them.

Both these sets of relations can in turn be quite complex and the finer details will be very case specific. A careful study of the relations between goods is required in each case to obtain the parametrisation of an EGT that best represents these production relations. We attempt to demonstrate this in our paper, where we focus on two important special cases with regards to the production relations forming (ii). The first, where each cleaning-up level helps jointly mitigate every emission type and the second, where cleaning-up activities are emission-specific and, while helping in mitigating some emissions, each can also increase other types of emissions. The paper also studies modelling of cases where a firm can be both a generator and a victim of its own emission. The model of an EGT of a firm whose emission has a negative effect on its intended output will exhibit Starrett [1972]-type non-convexities. The public-economic consequences of this will be less severe as it will be in the interest of the firm to internalise the negative effects that its own emission has on its intended output. The proposed models in this paper can be modified or extended appropriately to capture further features of emission generation and cleaning-up activities – *e.g.,* following Pethig [2006], we can extend the model to capture details of how the overall resources of a producing unit are partly devoted to abatement activities and model the forms into which abatement activities transform emissions.

In contrast to the parametrisation of the technology that is adopted in the output and weak-disposability based approaches, where a *single* distance function is derived either by radial expansions of emissions and intended outputs or by a direction vector that expands intended outputs and contracts emissions, an EGT that satisfies the axioms in this paper is parametrized by multiple distance functions, each capturing a distinct set of production relations that together compose our EGT. *E.g.,* the distance function that captures production relations in (i) is based on radial expansions of intended outputs and emissions. It is conditioned by the upper bounds on both intended outputs and emissions. The distance function that captures production relations in (ii) is based on radial contractions of intended outputs and emissions. It is conditioned by the lower bounds on emissions.

It is important in both applied and policy works to employ a model that best captures our intuitive understanding of relations between various variables of interest to increase the accuracy of results and policy prescriptions, which can vary significantly depending on the model employed. *E.g.,* employing a version of the model in this paper, MRL [2012] distinguish between, propose, and estimate technical and environmental efficiency indexes for a data set of 95 coal-fired plants in the U.S. They find significant differences between their estimates and those obtained from alternative models of EGTs and alternative efficiency indexes employed in the literature with the same data set.

The model of an EGT that is proposed in this paper can also provide fresh insights into applied and policy-oriented research. *E.g.,* we feel that the differential and distinct treatment of emission-causing and non-emission-causing goods and different cleaning-up activities permitted by our model of an EGT leads to novel definitions of the concept of marginal abatement cost (MAC). Our model permits several types of abatement strategies and the MAC will differ depending on the abatement strategy adopted. In one such on-going research agenda, we are employing this model to theoretically define MAC and measure it using the rich data that is now available from sources such as the International Energy Agency, the World Bank, and the Food and Agricultural Organization, *etc.* In particular, we will use the proposed model of EGT to estimate international differences in MAC and optimal abatement strategies, which will provide an empirical basis for international emission-trading policies and will identify the patterns of such trades.

APPENDIX

Proof of Theorem (BP):

(i) Let $y \in \mathcal{Y}(x_z, x_o, a)$. Under the maintained assumptions and (iv) of Remark 2, there exist $z \in \mathbf{R}_+^{m'}$ and $\langle \bar{y}, \bar{z} \rangle \in \mathbf{R}_+^{m+m'}$ such that $y \in P(x_z, x_o, a, z)$ and $\bar{y} \in P(\bar{x}_z, x_o, \bar{a}, \bar{z})$. Choose $\hat{z}$ such that $\hat{z} \geq z$ and $\hat{z} \geq \bar{z}$. Then (iii) of Remark 2 implies that $y \in P(x_z, x_o, a, \hat{z})$ and $\bar{y} \in P(\bar{x}_z, x_o, \bar{a}, \hat{z})$ and (CFD2x_z, a) implies that $P(x_z, x_o, a, \hat{z}) \subset P(\bar{x}_z, x_o, \bar{a}, \hat{z})$. Hence, $y \in P(\bar{x}_z, x_o, \bar{a}, \hat{z})$. (iv) of Remark 2 implies $y \in \mathcal{Y}(\bar{x}_z, x_o, \bar{a})$.

(ii) Let $\bar{z} \in Z(\bar{x}_z, x_o, \bar{a})$. Under the maintained assumptions, there exist $\bar{y} \in \mathbf{R}_+^{m}$ and $\langle y, z \rangle \in \mathbf{R}_+^{m+m'}$ such that $\bar{z} \in P(\bar{x}_z, x_o, \bar{a}, \bar{y})$ and $z \in P(x_z, x_o, a, y)$. Choose $\hat{y}$ such that $\hat{y} \leq y$ and $\hat{y} \leq \bar{y}$. Then (i) of Remark 2, (FDo), and (FDy_z) imply that $z \in P(x_z, x_o, a, \hat{y})$ and $\bar{z} \in P(\bar{x}_z, x_o, \bar{a}, \hat{y})$. If $x_z = \bar{x}_z = 0$ and $m_z = 0$, then Assumption (SD-JE$_{x_z, y_z}$) implies $P(\bar{x}_z, x_o, \bar{a}, \hat{y}) = P(x_z, x_o, a, \hat{y}) = 0$. Otherwise, (CCD$x_z, a$) implies that $P(\bar{x}_z, x_o, \bar{a}, \hat{y}) \subset P(x_z, x_o, a, \hat{y})$. Hence, in both cases, $\bar{z} \in P(x_z, x_o, a, \hat{y})$. Hence, $\bar{z} \in Z(x_z, x_o, a)$. ■

Proof of Lemma 1:

The conclusions are trivially true if $\mathcal{P}(x,a,z) = \emptyset$. If $\mathcal{P}(x,a,z) \neq \emptyset$, then (i) of Remark 2 implies $\mathcal{P}(x,a,z) \subseteq P(x,a,z)$. Suppose $y \in P(x,a,z)$. Then by definition of $P(x,a,z)$, there exists $z' \in \mathbf{R}^{m'}_+$ such that $z' \leq z$ and $y \in \mathcal{P}(x,a,z')$. If Assumption (IND$z$) holds then $\mathcal{P}(x,a,z) = \mathcal{P}(x,a,z')$. If Assumption (BEN$z$) holds then $\mathcal{P}(x,a,z') \subseteq \mathcal{P}(x,a,z)$. Either way, we have $y \in \mathcal{P}(x,a,z)$. Hence, $P(x,a,z) \subseteq \mathcal{P}(x,a,z)$. Conclusions of the lemma follow. ■

Proof of Theorem (IP):

(i) Suppose $\langle x,a,y,z\rangle \notin \mathbf{\Lambda_1}$. Then $\Lambda_1(x,a,y,z) = \emptyset$. So, by its definition, $D_1(x,a,y,z) = \infty$. Suppose $\langle x,a,y,z\rangle \in \mathbf{\Lambda_1}$. Then $\Lambda_1(x,a,y,z) \neq \emptyset$. Assumptions (BOUND$y$) and (C) imply that $\Lambda_1(x,a,y,z)$ is compact. Hence, $D_1(x,a,y,z)$ is well defined and unique.

(ii) $\langle x,a,y,z\rangle \in \mathcal{T}$ implies $1 \in \Lambda_1(x,a,y,z)$. Hence, from its definition, $D_1(x,a,y,z) \leq 1$.

(iii) Proofs of homogeneity of D_1 and convexity of D_1 if the mapping $\boldsymbol{\mathcal{P}} : \mathbf{\Omega} \mapsto \mathbf{R}^{m+m'}$ is convex valued:

- D_1 is homogeneous of degree one in y and z:

$$\begin{aligned}
D_1(x,a,\kappa y,\kappa z) &:= \inf \,\{\lambda_1 > 0 \mid \langle \frac{\kappa y}{\lambda_1}, \frac{\kappa z}{\lambda_1}\rangle \in \mathcal{P}(x,a)\} \\
&= \kappa \inf \,\{\frac{\lambda_1}{\kappa} > 0 \mid \langle \frac{y}{\frac{\lambda_1}{\kappa}}, \frac{z}{\frac{\lambda_1}{\kappa}}\rangle \in \mathcal{P}(x,a)\} \\
&= \kappa \; D_1(x,a,y,z)
\end{aligned}$$

- D_1 is convex in y and z: Need to show

$D_1(x,a,\alpha\bar{y} + (1-\alpha)\hat{y}, \alpha\bar{z} + (1-\alpha)\hat{z}) \leq \alpha D_1(x,a,\bar{y},\bar{z}) + (1-\alpha) D_1(x,a,\hat{y},\hat{z})$ for $\alpha \in [0,1]$. Let $\bar{\lambda}_1 = D_1(x,a,\bar{y},\bar{z})$ and $\hat{\lambda}_1 = D_1(x,a,\hat{y},\hat{z})$. The definition of D_1 implies $\frac{1}{\bar{\lambda}}\langle \bar{y},\bar{z}\rangle \in \mathcal{P}(x,a)$ and $\frac{1}{\hat{\lambda}}\langle \hat{y},\hat{z}\rangle \in \mathcal{P}(x,a)$. Define $\overset{*}{\alpha} = \frac{\alpha\bar{\lambda}_1}{\alpha\bar{\lambda}_1 + (1-\alpha)\hat{\lambda}_1}$. Then $(1-\overset{*}{\alpha}) = \frac{(1-\alpha)\hat{\lambda}_1}{\alpha\bar{\lambda}_1 + (1-\alpha)\hat{\lambda}_1}$ and $\overset{*}{\alpha} \in [0,1]$. Since $\mathcal{P}(x,a)$ is convex, this implies $\langle \overset{*}{\alpha}\frac{\bar{y}}{\bar{\lambda}_1} + (1-\overset{*}{\alpha})\frac{\hat{y}}{\hat{\lambda}_1},\ \overset{*}{\alpha}\frac{\bar{z}}{\bar{\lambda}_1} + (1-\overset{*}{\alpha})\frac{\hat{z}}{\hat{\lambda}_1}\rangle \in \mathcal{P}(x,a)$. But $\langle \overset{*}{\alpha}\frac{\bar{y}}{\bar{\lambda}_1} + (1-\overset{*}{\alpha})\frac{\hat{y}}{\hat{\lambda}_1},\ \overset{*}{\alpha}\frac{\bar{z}}{\bar{\lambda}_1} + (1-\overset{*}{\alpha})\frac{\hat{z}}{\hat{\lambda}_1}\rangle = \langle \frac{\alpha\bar{y} + (1-\alpha)\hat{y}}{\alpha\bar{\lambda}_1 + (1-\alpha)\hat{\lambda}_1},\ \frac{\alpha\bar{z} + (1-\alpha)\hat{z}}{\alpha\bar{\lambda}_1 + (1-\alpha)\hat{\lambda}_1}\rangle$. Hence, $\alpha\bar{\lambda}_1 + (1-\alpha)\hat{\lambda}_1 \in \Lambda_1(x,a,\alpha\bar{y} + (1-\alpha)\hat{y}, \alpha\bar{z} + (1-\alpha)\hat{z})$. Hence, $D_1(x,a,\alpha\bar{y} + (1-\alpha)\hat{y}, \alpha\bar{z} + (1-\alpha)\hat{z}) \leq \alpha\bar{\lambda}_1 + (1-\alpha)\hat{\lambda}_1 = \alpha D_1(x,a,\bar{y},\bar{z}) + (1-\alpha) D_1(x,a,\hat{y},\hat{z})$ for all $\alpha \in [0,1]$.

• D_1 is non-decreasing in y: Let $D_1(x,a,y,z) \equiv \lambda_1$, $\tilde{y} > y$ and $D_1(x,a,\tilde{y},z) \equiv \tilde{\lambda}_1$. Then, the definition of D_1 implies that $\langle \frac{\tilde{y}}{\tilde{\lambda}_1}, \frac{z}{\tilde{\lambda}_1}\rangle \in \mathcal{P}(x,a)$ and $\langle \frac{y}{\lambda_1}, \frac{z}{\lambda_1}\rangle \in \mathcal{P}(x,a)$. Clearly $\frac{y}{\tilde{\lambda}_1} < \frac{\tilde{y}}{\tilde{\lambda}_1}$. (FD$o$) and (FD$y_z$) imply $\langle \frac{y}{\tilde{\lambda}_1}, \frac{z}{\tilde{\lambda}_1}\rangle \in \mathcal{P}(x,a)$. Hence, $\tilde{\lambda}_1 \in \Lambda_1(x,a,y,z)$. Hence, $D_1(x,a,y,z) = \lambda_1 \leq \tilde{\lambda}_1 = D_1(x,a,\tilde{y},z)$.

• D_1 is non-increasing in x and non-decreasing in a: Let $\langle x,a\rangle \in \mathbf{\Omega}$, $\bar{x} \geq x$, and $\bar{a} \leq a$ such that $\langle \bar{x},\bar{a}\rangle \in \mathbf{\Omega}$ and $\langle \bar{x},\bar{a}\rangle \neq \langle x,a\rangle$. Two cases are possible when $\lambda_1 \equiv D_1(x,a,y,z)$ and $\bar{\lambda}_1 \equiv D_1(\bar{x},\bar{a},y,z)$ are well defined:

<u>Case 1.</u> $\frac{z}{\lambda_1} \in \mathcal{Z}(\bar{x},\bar{a})$: Hence, there exists $y' \in \mathbf{R}^m_+$ such that $\langle y', \frac{z}{\lambda_1}\rangle \in \mathcal{P}(\bar{x},\bar{a})$. Thus, $\mathcal{P}(\bar{x},\bar{a},\frac{z}{\lambda_1}) \neq \emptyset$. Assumption (CFD1$x_z,a$) implies $\mathcal{P}(x,a,\frac{z}{\lambda_1}) \subset \mathcal{P}(\bar{x},\bar{a},\frac{z}{\lambda_1})$. Hence, $\frac{y}{\lambda_1} \in \mathcal{P}(\bar{x},\bar{a},\frac{z}{\lambda_1})$. Hence, $\langle \frac{y}{\lambda_1}, \frac{z}{\lambda_1}\rangle \in \mathcal{P}(\bar{x},\bar{a})$. Hence, $\lambda_1 \in \Lambda_1(\bar{x},\bar{a},y,z)$. Hence, $D_1(\bar{x},\bar{a},y,z) \leq \lambda_1$.

<u>Case 2.</u> $\frac{z}{\lambda_1} \notin \mathcal{Z}(\bar{x},\bar{a})$: We claim that $\frac{z}{\lambda_1} \notin Z(\bar{x},\bar{a})$. Suppose not, *i.e.,* suppose $\frac{z}{\lambda_1} \in Z(\bar{x},\bar{a})$. Then since $Z(\bar{x},\bar{a}) \equiv \mathcal{Z}(\bar{x},\bar{a}) + \mathbf{R}^{m'}_+$ and $\frac{z}{\lambda_1} \notin \mathcal{Z}(\bar{x},\bar{a})$, we have $\frac{z}{\lambda_1} \notin W(\bar{x},\bar{a}) \equiv (\mathcal{Z}(\bar{x},\bar{a}) - \mathbf{R}^{m'}_+) \cap \mathbf{R}^{m'}_+$. But then $\frac{z}{\lambda_1} \in \mathcal{Z}(x,a) \subset W(x,a)$. Assumption (UB$z$) implies $W(x,a) \subset W(\bar{x},\bar{a})$. Hence, $\frac{z}{\lambda_1} \in W(\bar{x},\bar{a})$. This is a contradiction. Hence, $\frac{z}{\lambda_1} \notin Z(\bar{x},\bar{a})$. Since $\frac{z}{\lambda_1} \notin \mathcal{Z}(\bar{x},\bar{a})$ and $\frac{z}{\lambda_1} \in \mathcal{Z}(x,a) \subset W(x,a) \subset W(\bar{x},\bar{a})$ (by Assumption (UBz)), we have $\frac{z}{\lambda_1} \in W(\bar{x},\bar{a}) \setminus \mathcal{Z}(\bar{x},\bar{a})$. Recall, $\Lambda_1(\bar{x},\bar{a},y,z) \neq \emptyset$. Hence, there exists $\tilde{\lambda}_1 > 0$ such that $\frac{z}{\tilde{\lambda}_1} \in \mathcal{Z}(\bar{x},\bar{a})$. Since, $\frac{z}{\lambda_1} \in W(\bar{x},\bar{a}) \setminus \mathcal{Z}(\bar{x},\bar{a})$, it must be the case that $\tilde{\lambda}_1 < \lambda_1$, for otherwise Assumption (R1) implies that $\frac{z}{\tilde{\lambda}_1} \in W(\bar{x},\bar{a}) \setminus \mathcal{Z}(\bar{x},\bar{a})$. Hence, for any $\tilde{\lambda}_1 \in \Lambda_1(\bar{x},\bar{a},y,z)$, it must be the case that $\tilde{\lambda}_1 < \lambda_1$. Hence, $D_1(\bar{x},\bar{a},y,z) < \lambda_1$.

• D_1 is constant in z for all $\langle x,a,y,z\rangle \in \mathbf{\Lambda_1}$ such that $\frac{z}{D_1(x,a,y,z)} \in Int(\mathcal{Z}(x,a))$ if Assumption (INDz) holds: Choose $\tilde{z} \neq z$ such that $D_1(x,a,y,z) \equiv \lambda_1$ and $D_1(x,a,y,\tilde{z}) \equiv \tilde{\lambda}_1$ are well defined, $\frac{z}{\lambda_1} \in Int(\mathcal{Z}(x,a))$, and $\frac{\tilde{z}}{\tilde{\lambda}_1} \in Int(\mathcal{Z}(x,a))$. Suppose $\lambda_1 \neq \tilde{\lambda}_1$. WOLOG assume $\tilde{\lambda}_1 < \lambda_1$. Then $\frac{y}{\tilde{\lambda}_1} > \frac{y}{\lambda_1}$. Assumption (IND$z$) implies $\mathcal{P}(x,a,\frac{z}{\lambda_1}) = \mathcal{P}(x,a,\frac{z}{\tilde{\lambda}_1})$. Hence, $\frac{y}{\tilde{\lambda}_1} \in \mathcal{P}(x,a,\frac{z}{\lambda_1})$. Since $\frac{z}{\lambda_1} \in Int(\mathcal{Z}(x,a))$, there exists $\epsilon \in (0,\lambda_1)$ such that $\frac{z}{\kappa} \in \mathcal{Z}(x,a)$ whenever $\epsilon \leq \kappa \leq \lambda_1$. Since $\frac{y}{\tilde{\lambda}_1} \in \mathcal{P}(x,a,\frac{z}{\lambda_1})$, Assumptions (FD$o$) and (FD$y_z$) imply that $\frac{y}{\kappa} \in \mathcal{P}(x,a,\frac{z}{\lambda_1})$ whenever $\tilde{\lambda}_1 \leq \kappa \leq \lambda_1$. Choose κ such that $\max\{\epsilon, \tilde{\lambda}_1\} \leq \kappa < \lambda_1$. Then $\frac{z}{\kappa} \in \mathcal{Z}(x,a)$ and $\frac{y}{\kappa} \in \mathcal{P}(x,a,\frac{z}{\lambda_1})$. Assumption (IND$z$) implies $\frac{y}{\kappa} \in \mathcal{P}(x,a,\frac{z}{\kappa})$. So $\kappa \in \Lambda_1(x,a,y,z)$.

But $\kappa < \lambda_1 \equiv D_1(x,a,y,z)$. This is a contradiction to the definition of $D_1(x,a,y,z)$. Hence, $\lambda_1 = \tilde{\lambda}_1$.

• D_1 is non-decreasing in z if Assumption (DETz) holds: Choose $\tilde{z} > z$ such that $D_1(x,a,y,z) \equiv \lambda_1$ and $D_1(x,a,y,\tilde{z}) \equiv \tilde{\lambda}_1$ are well defined. The definition of D_1 implies that $\frac{y}{\tilde{\lambda}_1} \in \mathcal{P}(x,a,\frac{\tilde{z}}{\tilde{\lambda}_1})$. $\frac{z}{\tilde{\lambda}_1} < \frac{\tilde{z}}{\tilde{\lambda}_1}$. Two cases arise:

<u>Case 1.</u> $\mathcal{P}(x,a,\frac{z}{\tilde{\lambda}_1}) \neq \emptyset$: Assumption (DET$z$) implies $\mathcal{P}(x,a,\frac{\tilde{z}}{\tilde{\lambda}_1}) \subset \mathcal{P}(x,a,\frac{z}{\tilde{\lambda}_1})$. Hence, $\langle\frac{y}{\tilde{\lambda}_1},\frac{z}{\tilde{\lambda}_1}\rangle \in \mathcal{P}(x,a)$. Hence, $\tilde{\lambda}_1 \in \Lambda_1(x,a,y,z)$. Hence, $\lambda_1 \leq \tilde{\lambda}_1$.

<u>Case 2.</u> $\mathcal{P}(x,a,\frac{z}{\tilde{\lambda}_1}) = \emptyset$: This implies $\langle\frac{y}{\tilde{\lambda}_1},\frac{z}{\tilde{\lambda}_1}\rangle \notin \mathcal{P}(x,a)$ and $\frac{z}{\tilde{\lambda}_1} \notin \mathcal{Z}(x,a)$. But since $\frac{z}{\tilde{\lambda}_1} < \frac{\tilde{z}}{\tilde{\lambda}_1}$ and $\frac{\tilde{z}}{\tilde{\lambda}_1} \in \mathcal{Z}(x,a)$, we have $\frac{z}{\tilde{\lambda}_1} \in W(x,a) \setminus \mathcal{Z}(x,a)$. Assumption (R1) implies that for all $\kappa \geq \tilde{\lambda}_1$, we have $\kappa \notin \Lambda_1(x,a,y,z)$. Hence, if $\kappa \in \Lambda_1(x,a,y,z)$, then $\kappa < \tilde{\lambda}_1$. Hence, $\lambda_1 \equiv D_1(x,a,y,z) < \tilde{\lambda}_1$.

• D_1 is jointly continuous in its arguments if Assumption (INDz) or (DETz) holds: Define the mapping $\bar{\boldsymbol{\mathcal{P}}} : \boldsymbol{\Omega} \mapsto \mathbf{R}_+^{m+m'}$ with image $\bar{\boldsymbol{\mathcal{P}}}(x,a) \equiv \big(\mathcal{P}(x,a) - \mathbf{R}_+^{m+m'}\big) \cap \mathbf{R}_+^{m+m'}$, the mapping $\bar{D}_1 : \boldsymbol{\Lambda_1} \longrightarrow \mathbf{R}_+$ with image $\bar{D}_1(x,a,y,z) \equiv \inf\,\{\lambda_1 > 0 \mid \langle\frac{y}{\lambda_1},\frac{z}{\lambda_1}\rangle \in \bar{\boldsymbol{\mathcal{P}}}(x,a)\}$, and the mapping $\bar{\Lambda}_1 : \boldsymbol{\Lambda_1} \mapsto \mathbf{R}_+$ with image $\bar{\Lambda}_1(x,a,y,z) = \{\lambda_1 > 0 \mid \langle\frac{y}{\lambda_1},\frac{z}{\lambda_1}\rangle \in \bar{\boldsymbol{\mathcal{P}}}(x,a)\}$.

<u>Step 1:</u> We show that for all $\langle x,a,y,z\rangle \in \boldsymbol{\Lambda_1}$, $\bar{D}_1(x,a,y,z) = D_1(x,a,y,z)$:

Suppose there exists $\langle x,a,y,z\rangle \in \boldsymbol{\Lambda_1}$ such that $\bar{\lambda}_1 \equiv \bar{D}_1(x,a,y,z) \neq D_1(x,a,y,z) \equiv \lambda_1$. Then $\langle\frac{y}{\bar{\lambda}_1},\frac{z}{\bar{\lambda}_1}\rangle \in \bar{\boldsymbol{\mathcal{P}}}(x,a)$. Since $\mathcal{P}(x,a) \subset \bar{\boldsymbol{\mathcal{P}}}(x,a)$, we have $\lambda_1 > \bar{\lambda}_1$ and $\langle\frac{y}{\bar{\lambda}_1},\frac{z}{\bar{\lambda}_1}\rangle \notin \mathcal{P}(x,a)$. Two cases are possible: $\mathcal{P}(x,a,\frac{z}{\bar{\lambda}_1}) = \emptyset$ and $\mathcal{P}(x,a,\frac{z}{\bar{\lambda}_1}) \neq \emptyset$.

Case 1: Suppose $\mathcal{P}(x,a,\frac{z}{\bar{\lambda}_1}) = \emptyset$. Then there exists no $\tilde{y} \geq 0$ such that $\tilde{y} \in \mathcal{P}(x,a,\frac{z}{\bar{\lambda}_1})$. Hence, $\frac{z}{\bar{\lambda}_1} \notin \mathcal{Z}(x,a)$. But by definition of $\bar{\boldsymbol{\mathcal{P}}}(x,a)$, there exists $\langle\tilde{y},\tilde{z}\rangle \in \mathcal{P}(x,a)$ such that $\langle\tilde{y},\tilde{z}\rangle > \langle\frac{y}{\bar{\lambda}_1},\frac{z}{\bar{\lambda}_1}\rangle$. Hence, $\tilde{z} \in \mathcal{Z}(x,a)$ and $\frac{z}{\bar{\lambda}_1} \leq \tilde{z}$. Hence, by definition of $W(x,a)$, we have $\frac{z}{\bar{\lambda}_1} \in W(x,a)$. Hence, Assumption (R1) implies that for all $\kappa > 0$ such that $\langle\frac{y}{\kappa\bar{\lambda}_1},\frac{z}{\kappa\bar{\lambda}_1}\rangle \in \mathcal{P}(x,a)$, it must be the case that $0 < \kappa < 1$. Hence, if $\bar{\kappa} > 0$ is such that $\lambda_1 = \bar{\kappa}\bar{\lambda}_1$, then $\bar{\kappa} < 1$ and hence, $\lambda_1 < \bar{\lambda}_1$, which is a contradiction to our assumption that $\lambda_1 \neq \bar{\lambda}_1$, which implies $\lambda_1 > \bar{\lambda}_1$.

Case 2: Suppose $\mathcal{P}(x,a,\frac{z}{\bar{\lambda}_1}) \neq \emptyset$. Suppose Assumption (INDz) is true. Then $\mathcal{P}(x,a,\frac{z}{\bar{\lambda}_1}) = \mathcal{P}(x,a,\frac{z}{\lambda_1}) = \mathcal{P}(x,a,\hat{z})$ for all $\hat{z}$ such that $\mathcal{P}(x,a,\hat{z}) \neq \emptyset$. Suppose Assumption (DETz) is true. Then $\mathcal{P}(x,a,\hat{z}) \subset \mathcal{P}(x,a,\frac{z}{\lambda_1}) \subset \mathcal{P}(x,a,\frac{z}{\bar{\lambda}_1})$ for all $\hat{z} > \frac{z}{\bar{\lambda}_1}$ such that $\mathcal{P}(x,a,\hat{z}) \neq \emptyset$. $\frac{y}{\bar{\lambda}_1} \in \bar{\boldsymbol{\mathcal{P}}}(x,a,\frac{z}{\bar{\lambda}_1})$ implies by definition of $\bar{\boldsymbol{\mathcal{P}}}$ that there exists $\langle \hat{y}, \hat{z} \rangle > \langle \frac{y}{\lambda_1}, \frac{z}{\lambda_1} \rangle$ such that $\hat{y} \in \mathcal{P}(x, a, \hat{z})$. Hence, Assumptions (INDz) and (DETz) imply $\hat{y} \in \mathcal{P}(x, a, \frac{z}{\lambda_1})$. Since $\frac{y}{\lambda_1} \leq \hat{y}$, Assumptions (FD$o$) and (FD$y_z$) imply that $\frac{y}{\lambda_1} \in \mathcal{P}(x, a, \frac{z}{\lambda_1})$. This contradicts $\frac{y}{\lambda_1} \notin \mathcal{P}(x, a, \frac{z}{\lambda_1})$.

Cases (1) and (2) imply that $\lambda_1 = \bar{\lambda}_1$.

Step 2: We show that the mapping $\bar{\Lambda}_1$ is upper hemi-continuous: This follows from Assumption (C) which implies upper hemi-continuity of the mapping $\boldsymbol{\mathcal{P}}$, and hence the upper hemi-continuity of the mapping $\bar{\boldsymbol{\mathcal{P}}}$.

Step 3: We show that the mapping $\bar{\Lambda}_1$ is lower hemi-continuous:

Let $\{\langle x^v, a^v, y^v, z^v \rangle\} \longrightarrow \langle \bar{x}, \bar{a}, \bar{y}, \bar{z} \rangle$ such that $\langle x^v, a^v, y^v, z^v \rangle \in \boldsymbol{\Lambda_1}$ for all v. Assumption (C) implies $\boldsymbol{\Lambda_1}$ is closed and hence $\langle \bar{x}, \bar{a}, \bar{y}, \bar{z} \rangle \in \boldsymbol{\Lambda_1}$. Suppose $\bar{\lambda}_1 \in \bar{\Lambda}_1(\bar{x}, \bar{a}, \bar{y}, \bar{z})$. Then, $\langle \frac{\bar{y}}{\bar{\lambda}_1}, \frac{\bar{z}}{\bar{\lambda}_1} \rangle \in \bar{\boldsymbol{\mathcal{P}}}(\bar{x}, \bar{a})$. We need to show that there exists a sequence $\{\lambda_1^v\} \longrightarrow \bar{\lambda}_1$ such that $\lambda_1^v \in \bar{\Lambda}_1(x^v, a^v, y^v, z^v)$ for all big enough v. Since $\bar{\boldsymbol{\mathcal{P}}} : \boldsymbol{\Lambda_1} \mapsto \mathbf{R}_+^{m+m'}$ is a continuous map, there exists $\{\langle \hat{y}^v, \hat{z}^v \rangle\} \longrightarrow \langle \frac{\bar{y}}{\bar{\lambda}_1}, \frac{\bar{z}}{\bar{\lambda}_1} \rangle$ such that $\langle \hat{y}^v, \hat{z}^v \rangle \in \bar{\boldsymbol{\mathcal{P}}}(x^v, a^v)$ for all big enough v. Define $\mathcal{Q}^v = \{\langle y, z \rangle \in \mathbf{R}_+^{m+m'} \mid \langle y, z \rangle \leq \langle \hat{y}^v, \hat{z}^v \rangle\}$. Choose a sequence of scalars $\{\epsilon^v\} \longrightarrow 0$ such that $\epsilon^v > 0$ and $\parallel \langle \hat{y}^v, \hat{z}^v \rangle \parallel - \epsilon^v > 0$. For every v choose λ_1^v such that $\parallel \langle \hat{y}^v, \hat{z}^v \rangle \parallel - \epsilon^v \leq \parallel \langle \frac{y^v}{\lambda_1^v}, \frac{z^v}{\lambda_1^v} \rangle \parallel \leq \parallel \langle \hat{y}^v, \hat{z}^v \rangle \parallel$ and $\langle \frac{y^v}{\lambda_1^v}, \frac{z^v}{\lambda_1^v} \rangle \in \mathcal{Q}^v$. Since $\langle \hat{y}^v, \hat{z}^v \rangle \in \bar{\boldsymbol{\mathcal{P}}}(x^v, a^v)$, this implies $\langle \frac{y^v}{\lambda_1^v}, \frac{z^v}{\lambda_1^v} \rangle \in \bar{\boldsymbol{\mathcal{P}}}(x^v, a^v)$. Hence, $\lambda_1^v \in \bar{\Lambda}_1(x^v, a^v, y^v, z^v)$ for all v. As $v \longrightarrow \infty$, we have $\{\langle \hat{y}^v, \hat{z}^v \rangle\} \longrightarrow \langle \frac{\bar{y}}{\bar{\lambda}_1}, \frac{\bar{z}}{\bar{\lambda}_1} \rangle$ and $\{\parallel \langle \hat{y}^v, \hat{z}^v \rangle \parallel - \epsilon^v\} \longrightarrow \parallel \langle \frac{\bar{y}}{\bar{\lambda}_1}, \frac{\bar{z}}{\bar{\lambda}_1} \rangle \parallel$. Hence, $\{\langle \frac{y^v}{\lambda_1^v}, \frac{z^v}{\lambda_1^v} \rangle\} \longrightarrow \langle \frac{\bar{y}}{\bar{\lambda}_1}, \frac{\bar{z}}{\bar{\lambda}_1} \rangle$. Hence, since $\{\langle y^v, z^v \rangle\} \longrightarrow \langle \bar{y}, \bar{z} \rangle$, we have $\{\lambda_1^v\} \longrightarrow \bar{\lambda}_1$.

Step 4: Steps (2) and (3) imply the mapping $\bar{\Lambda}_1$ is continuous. Hence, from the Berge's Theorem of the Maximum, $\bar{D}_1$, restricted to the domain $\boldsymbol{\Lambda_1}$, is jointly continuous in all its arguments. Joint continuity of D_1, restricted to the domain $\boldsymbol{\Lambda_1}$, follows as $D_1 = \bar{D}_1$. ■

Proof of Theorem (EG):

(i) Suppose $\langle x, a, y, z \rangle \notin \boldsymbol{\Lambda_2}$. Then $\Lambda_2(x, a, y, z) = \emptyset$. So, by its definition, $D_2(x, a, y, z) = \infty$. Suppose $\langle x, a, y, z \rangle \in \boldsymbol{\Lambda_2}$. Then $\Lambda_2(x, a, y, z) \neq \emptyset$. $P(x, a) \subset \mathbf{R}_+^{m+m'}$ is bounded from below. Hence, 0 is a lower bound of the set $\Lambda_2(x, a, y, z)$. Hence, $\Lambda_2(x, a, y, z)$ has a greatest lower bound. Assumption (C) and the definition of T imply T is a closed set. Hence, $\Lambda_2(x,a,y,z)$ is a closed set. Hence, $\Lambda_2(x,a,y,z)$ contains its greatest lower bound. Hence, $D_2(x,a,y,z)$ is well defined.

(ii) $\langle x,a,y,z\rangle \in T \implies \langle y,z\rangle \in P(x,a)$. Hence, $1 \in \Lambda_2(x,a,y,z)$ and $D_2(x,a,y,z) \leq 1$.

(iii) Proofs of linear-homogeneity and convexity of D_2 in y and z are similar to Part (iii) of Theorem (IP).

• D_2 is non-increasing in z and non-decreasing in y: Let $D_2(x,a,y,z) = \lambda_2$, $\bar{y} \leq y$, $\bar{z} \geq z$, and $D_2(x,a,\bar{y},\bar{z}) = \bar{\lambda}_2$. From the definition of D_2 it follows that $\langle \lambda_2 y, \lambda_2 z\rangle \in P(x,a)$. (FD$y_z$), (FD$y_o$), and Parts (i) and (iii) of Remark 2 imply that $\langle \lambda_2\bar{y}, \lambda_2\bar{z}\rangle \in P(x,a)$. Hence, $\lambda_2 \in \Lambda_2(x,a,\bar{y},\bar{z})$. Hence, $\bar{\lambda}_2 \leq \lambda_2$.

• D_2 is non-decreasing in x and non-increasing in a if Assumption (CFD2x_z,a) holds: Let $D_2(x,a,y,z) \equiv \lambda_2$. Let $\bar{x} \geq x$ and $\bar{a} \leq a$ such that $\langle \bar{x},\bar{a}\rangle \in \mathbf{\Omega}$ and $\langle \bar{x},\bar{a}\rangle \neq \langle x,a\rangle$. Two cases are possible when $\lambda_2 \equiv D_2(x,a,y,z)$ and $\bar{\lambda}_2 \equiv D_2(\bar{x},\bar{a},y,z)$ are well defined:

<u>Case 1.</u> $\bar{\lambda}_2 y \in \mathcal{Y}(x,a)$: Hence there exists $z' \in \mathbf{R}_+^{m'}$ such that $\langle \bar{\lambda}_2 y, z'\rangle \in P(x,a)$. Hence, $P(x,a,\bar{\lambda}_2 y) \neq \emptyset$. Also, $\bar{\lambda}_2 z \in P(\bar{x},\bar{a},\bar{\lambda}_2 y)$. If $x_z = \bar{x}_z = 0$ and $m_z = 0$, then Assumption (SD-JE$_{x_z,y_z}$) implies $P(\bar{x},\bar{a},\bar{\lambda}_2 y) = P(x,a,\bar{\lambda}_2 y) = 0$. Otherwise, Assumption (CCDx_z,a) implies that $P(\bar{x},\bar{a},\bar{\lambda}_2 y) \subset P(x,a,\bar{\lambda}_2 y)$. Hence, in both cases, $\langle \bar{\lambda}_2 y, \bar{\lambda}_2 z\rangle \in P(x,a)$. Hence, $\bar{\lambda}_2 \in \Lambda_2(x,a,y,z)$. Hence, $D_2(x,a,y,z) \leq \bar{\lambda}_2$.

<u>Case 2.</u> $\bar{\lambda}_2 y \notin \mathcal{Y}(x,a)$: Assumption (CFD2$x_z,a$) implies that Conclusion (i) of Theorem (BP) holds. Hence, $\mathcal{Y}(x,a) \subset \mathcal{Y}(\bar{x},\bar{a})$. (FD$o$) and (FD$y_z$) imply that $\mathcal{Y}(x,a) = \mathcal{Y}(x,a) - \mathbf{R}_+^m$. Hence, for any $\kappa > 0$ such that $\kappa y \in \mathcal{Y}(x,a)$, it must be the case that $\kappa < \bar{\lambda}_2$. (Such a κ exists as $\Lambda_2(x,a,y,z) \neq \emptyset$.) Hence, $\kappa < \bar{\lambda}_2$ for all $\kappa \in \Lambda_2(x,a,y,z)$. Hence, $D_2(x,a,y,z) < \bar{\lambda}_2$.

• D_2 is constant in y_o and x_o:

<u>Case 1.</u> D_2 is constant in x_o: This follows because Assumption (INDo) implies that $P(x_z,x_o,a) = P(x_z,\bar{x}_o,a)$ for all $\langle x_z,x_o,a\rangle$ and $\langle x_z,\bar{x}_o,a\rangle$ in $\mathbf{\Omega}$. Hence, $\Lambda_2(x_z,x_o,a,y,z) = \Lambda_2(x_z,\bar{x}_o,a,y,z)$ whenever $\langle x_z,x_o,a,y,z\rangle$ and $\langle x_z,\bar{x}_o,a,y,z\rangle$ in $\mathbf{\Lambda_2}$.

<u>Case 2.</u> D_2 is constant in y_o: Suppose $\langle x,a,y_z,y_o,z\rangle$ and $\langle x,a,y_z,\bar{y}_o,z\rangle$ in $\mathbf{\Lambda_2}$ and $\lambda_2 \equiv D_2(x,a,y_z,y_o,z) \neq D_2(x,a,y_z,\bar{y}_o,z) \equiv \bar{\lambda}_2$. WOLOG assume that $\bar{\lambda}_2 < \lambda_2$. Hence, $\bar{\lambda}_2 z < \lambda_2 z$ and $\bar{\lambda}_2 y_z < \lambda_2 y_z$. Assumptions (INDo) and (FDy_z) imply $P(x,a,\bar{\lambda}_2 y_z,\bar{\lambda}_2\bar{y}_o) = P(x,a,\lambda_2 y_z,\lambda_2 y_o)$. Hence, $\bar{\lambda}_2 z \in P(x,a,\lambda_2 y_z,\lambda_2 y_o)$. Hence,

$\langle \lambda_2 y, \bar{\lambda}_2 z\rangle \in P(x, a,)$. Since $\bar{\lambda}_2 < \lambda_2$, Assumptions (FD$o$) and (FD$y_z$) imply $\langle \bar{\lambda}_2 y, \bar{\lambda}_2 z\rangle \in P(x, a)$. But then, $\bar{\lambda}_2 \in \Lambda_2(x, a, y_z, y_o, z)$ and $\bar{\lambda}_2 < \lambda_2 \equiv D_2(x, a, y_z, y_o, z)$. This is a contradiction to the definition of $D_2(x, a, y_z, y_o, z)$. Hence, $\bar{\lambda}_2 = \lambda_2$.

• D_2 is jointly continuous in its arguments if $m_z = 0$: Proof is similar to proof of joint continuity of D_1 in Theorem (IP). Detailed steps of the proof can be found in the working paper version Murty [2012]. ■

Proof of Theorem (BP-REPR):

It is clear that (5.7) follows from (7.3) and the definitions of $\mathcal{T}_1$ and T_2 in (5.4) and (7.2). We prove (7.3) below.

$\Longrightarrow$ of (7.3): Follows directly from Parts (ii) of Theorems (IP) and (EG).

$\Longleftarrow$ of (7.3): We show that $\langle x, a, y, z\rangle \notin \mathcal{T}$ implies $D_1(x, a, y, z) > 1$ or $D_2(x, a, y, z) > 1$.

- Suppose $\langle x, a, y, z\rangle \in \mathbf{\Omega}\times\mathbf{R}_+^{m+m'}$ is such that $\lambda_2 \equiv D_2(x, a, y, z) \leq 1$ and $\langle x, a, y, z\rangle \notin \mathcal{T}$. We show that this implies $D_1(x, a, y, z) > 1$. By definition of D_2, $\langle \lambda_2 y, \lambda_2 z\rangle \in P(x, a)$.

  Case 1. $\langle \lambda_2 y, \lambda_2 z\rangle \in \mathcal{P}(x, a)$: This implies $\lambda_2 \neq 1$, for otherwise $\langle x, a, y, z\rangle \in \mathcal{T}$, which is a contradiction to the maintained assumption. Hence, $\lambda_2 < 1$. We claim that if $\kappa > 0$ is such that $\frac{1}{\kappa}\langle y, z\rangle \in \mathcal{P}(x, a)$, then $\kappa > 1$. Suppose not. Then there exists $\kappa \in (0, 1]$ such that $\frac{1}{\kappa}\langle y, z\rangle \in \mathcal{P}(x, a)$. Note that $\kappa \neq 1$, for otherwise $\langle x, a, y, z\rangle \in \mathcal{T}$, which is a contradiction to the maintained assumptions. Hence, $\lambda_2\langle y, z\rangle < \langle y, z\rangle < \frac{1}{\kappa}\langle y, z\rangle$. Since $\mathcal{P}(x, a)$ is convex, $\langle y, z\rangle$ can be written as a convex combination of $\lambda_2\langle y, z\rangle$ and $\frac{1}{\kappa}\langle y, z\rangle$. Hence, $\langle y, z\rangle \in \mathcal{P}(x, a)$, which is a contradiction. Hence, $\kappa > 1$ for all $\kappa \in \Lambda_1(x, a, y, z)$. Hence $D_1(x, a, y, z) > 1$.

  Case 2. $\langle \lambda_2 y, \lambda_2 z\rangle \notin \mathcal{P}(x, a)$: Combined with Assumption (R2) and the fact that $\langle \lambda_2 y, \lambda_2 z\rangle \in P(x, a)$, we have $\kappa\langle \lambda_2 y, \lambda_2 z\rangle \notin \mathcal{P}(x, a)$ for all $\kappa \geq 1$. Further, if there exists $\kappa \in [0, 1)$ such that $\kappa\langle \lambda_2 y, \lambda_2 z\rangle \in \mathcal{P}(x, a)$, then Part (i) of Remark 2 implies $\kappa\langle \lambda_2 y, \lambda_2 z\rangle \in P(x, a)$. Hence, $\kappa\lambda_2 \in \Lambda_1(x, a, y, z)$ and $\kappa\lambda_2 < \lambda_2$. This is a contradiction to $\lambda_2 = D_2(x, a, y, z)$. Hence, there exists no $\kappa > 0$ such that $\frac{1}{\kappa}\langle y, z\rangle \in \mathcal{P}(x, a)$. Hence, $\Lambda_1(x, a, y, z) = \emptyset$, so that $D_1(x, a, y, z) = \infty > 1$.

- Suppose $\langle x, a, y, z\rangle \in \mathbf{\Omega}\times\mathbf{R}_+^{m+m'}$ is such that $\lambda_1 \equiv D_1(x, a, y, z) \leq 1$ and $\langle x, a, y, z\rangle \notin \mathcal{T}$. We show that this implies $D_2(x, a, y, z) > 1$. By definition of D_1, $\langle \frac{y}{\lambda_1}, \frac{z}{\lambda_1}\rangle \in \mathcal{P}(x, a)$. Hence, from our maintained assumptions, it follows that $\lambda_1 \neq 1$, so that

$\lambda_1 < 1$ and $\langle y, z\rangle < \frac{1}{\lambda_1}\langle y, z\rangle$. Let $\lambda_2 \equiv D_2(x, a, y, z)$.

Case 1. $\langle \lambda_2 y, \lambda_2 z\rangle \in \mathcal{P}(x, a)$: Note, $\lambda_2 \neq 1$. For otherwise, $\langle x, a, y, z\rangle \in \mathcal{T}$, which is a contradiction to the maintained assumption. Suppose $0 \leq \lambda_2 < 1$. Then $\lambda_2\langle y, z\rangle < \langle y, z\rangle < \frac{1}{\lambda_1}\langle y, z\rangle$. Then convexity of $\mathcal{P}(x, a)$ implies that $\langle y, z\rangle$ is a convex combination of $\lambda_2\langle y, z\rangle$ and $\frac{1}{\lambda_1}\langle y, z\rangle$ and hence is in $\mathcal{P}(x, a)$, which is a contradiction.

Case 2. $\langle \lambda_2 y, \lambda_2 z\rangle \notin \mathcal{P}(x, a)$: Then Assumption (R2), $\lambda_1 < 1$, $\langle \lambda_2 y, \lambda_2 z\rangle \in P(x, a)$, and $\frac{1}{\lambda_1}\langle y, z\rangle \in \mathcal{P}(x, a)$ imply that $1 < \frac{1}{\lambda_1} < \lambda_2$. Hence, $\lambda_2 > 1$. ■

Proof of Theorem (BP-EFF):

Suppose at least one of $D_1(x, a, y, z)$ or $D_2(x, a, y, z)$ is not equal to 1. Three possibilities arise:

Case 1: At least one of $D_1(x, a, y, z)$ or $D_2(x, a, y, z)$ is greater than 1: Theorem (BP-REPR) implies $\langle x, a, y, z\rangle \notin \mathcal{T}$ and hence $\langle y, z\rangle$ is not a strictly efficient point of $\mathcal{P}(x, a)$.

Case 2: Suppose $\lambda_1 \equiv D_1(x, a, y, z) < 1$ and $D_2(x, a, y, z) \leq 1$: Theorem (BP-REPR) implies $\langle x, a, y, z\rangle \in \mathcal{T}$. $\frac{y}{\lambda_1} > y$ and $\frac{z}{\lambda_1} > z$. If Assumption (INDz) is true, then $\mathcal{P}(x, a, z) = \mathcal{P}(x, a, \frac{z}{\lambda_1})$. If Assumption (DET$z$) is true, then $\mathcal{P}(x, a, \frac{z}{\lambda_1}) \subset \mathcal{P}(x, a, z)$. Hence, in either case, $\langle \frac{y}{\lambda_1}, z\rangle \in \mathcal{P}(x, a)$. Hence, $\langle y, z\rangle$ is not a strictly efficient point of $\mathcal{P}(x, a)$.

Case 3: Suppose $\lambda_2 \equiv D_2(x, a, y, z) < 1$ and $D_1(x, a, y, z) \leq 1$: Theorem (BP-REPR) implies $\langle x, a, y, z\rangle \in \mathcal{T}$. $\lambda_2 y < y$ and $\lambda_2 z < z$. Since $\mathcal{T}$ is a BPT, Assumption (INDo) is true. Combined with the assumption that $m_z = 0$, this implies $\mathcal{P}(x, a, y) = \mathcal{P}(x, a, \lambda_2 y)$. Hence, $\langle y, \lambda_2 z\rangle \in \mathcal{P}(x, a)$. Hence, $\langle y, z\rangle$ is not a strictly efficient point of $\mathcal{P}(x, a)$. ■

Proof of Theorem (EG-ESCU):

(i) Suppose $\langle x, a, y\rangle \notin \mathbf{\Theta}$. Then $\Gamma_k(x, a, y, z_k) = \emptyset$. So, by its definition, $E_k(x, a, y, z_k) = \infty$. Suppose $z_k = 0$ and $0 \notin Z_k(x, a, y)$. Then too $\Gamma_k(x, a, y, z_k) = \emptyset$ and $E_k(x, a, y, z_k) = \infty$. Suppose $\langle x, a, y\rangle \in \mathbf{\Theta}$ and $z_k > 0$. Then $Z_k(x, a, y) \neq \emptyset$. Let $\hat{z}_k \in Z_k(x, a, y)$. Then $\gamma_k = \frac{\hat{z}_k}{z_k} \in \Gamma_k(x, a, y, z_k)$. Hence, $\Gamma_k(x, a, y, z_k) \neq \emptyset$. By its definition, 0 is a lower bound for $\Gamma_k(x, a, y, z_k)$. Hence, $\Gamma_k(x, a, y, z_k)$ has a greatest lower bound. Hence, $E_k(x, a, y, z_k)$ is well defined. Suppose $\langle x, a, y\rangle \in \mathbf{\Theta}$ and $0 \in Z_k(x, a, y)$. Then $0 \in \Gamma_k(x, a, y, z_k)$ for all $z_k \geq 0$. Hence, $E_k(x, a, y, z_k)$ is zero and well defined.

(ii) $\langle x, a, y, z\rangle \in T \implies z_k \in Z(x, a, y)$. Hence, $1 \in \Gamma_k(x, a, y, z_k)$ and $E_k(x, a, y, z_k) \leq 1$.

(iii) Proofs of linear-homogeneity and convexity of E_k in z are similar to Part (iii) of Theorem (IP) once we note that the set $Z_k(x, a, y)$ is a closed and convex interval.

- E_k is non-increasing in z_k: Let $E_k(x, a, y, z_k) = \gamma_k$, $\bar{z}_k > z_k$, and $E_k(x, a, y, \bar{z}_k) = \bar{\gamma}_k$. From the definition of E_k it follows that $\gamma_k z_k \in Z_k(x, a, y)$. Part (iii) of Remark 2 implies that $\gamma_k \bar{z}_k \in Z_k(x, a, y)$. Hence, $\gamma_k \in \Gamma_k(x, a, y, \bar{z}_k)$. Hence, $\bar{\gamma}_k \leq \gamma_k$.
- E_k is non-decreasing in y: Let $E_k(x, a, y, z_k) = \gamma_k$, $\bar{y} \leq y$, and $E_k(x, a, \bar{y}, z_k) = \bar{\gamma}_k$. From the definition of E_k it follows that $\gamma_k z_k \in Z_k(x, a, y)$. (FD$y_z$), (FD$y_o$), and Part (i) of Remark 2 imply that $\gamma_2 z_k \in Z_k(x, a, \bar{y})$. Hence, $\gamma_k \in \Gamma_k(x, a, \bar{y}, z)$. Hence, $\bar{\gamma}_k \leq \gamma_k$.
- E_k is non-decreasing in x_z: Let $\bar{x}_z \geq x_z$, $E_k(x_z, x_o, a, y, z_k) \equiv \gamma_2$, and $E_k(\bar{x}_z, x_o, a, y, z_k) \equiv \bar{\gamma}_2$. Since, we are operating in the domain where E_k is well-defined, we have $P(x_z, x_o, a, y) \neq \emptyset$ and $P(\bar{x}_z, x_o, a, y) \neq \emptyset$. Hence, Assumption (CCD$x_z, a$) implies that $P(\bar{x}_z, x_o, a, y) \subset P(x_z, x_o, a, y)$. Assumption (EG-IND$z$) and Remark 3 imply that $Z_1(\bar{x}_z, x_o, a, y) \times Z_2(\bar{x}_z, x_o, a, y) \subset Z_1(x_z, x_o, a, y) \times Z_2(x_z, x_o, a, y)$. Hence, $Z_1(\bar{x}_z, x_o, a, y) \subset Z_1(x_z, x_o, a, y)$. Hence, since $\bar{\gamma}_k z_k \in Z_k(\bar{x}_z, x_o, a, y)$, we also have $\bar{\gamma}_k z_k \in Z_k(x_z, x_o, a, y)$. Thus, $\bar{\gamma}_k \in \Gamma_k(x_z, x_o, a, y, z_k)$. Hence, $\gamma_k \leq \bar{\gamma}_k$.
- E_k is non-increasing in a_k: Let $\bar{a}_k \geq a_k$, $E_k(x, a_k, a_{-k}, y, z_k) \equiv \gamma_2$, and $E_k(x, \bar{a}_k, a_{-k}, y, z_k) \equiv \bar{\gamma}_2$. Since, we are operating in the domain where E_k is well-defined, we have $P(x, a_k, a_{-k}, y) \neq \emptyset$ and $P(x, \bar{a}_k, a_{-k}, y) \neq \emptyset$. Hence, Assumption (ESCU) implies that $Z_k(x, a_k, a_{-k}, y) \subset Z_k(x, \bar{a}_k, a_{-k}, y)$. Hence, since $\gamma_k z_k \in Z_k(x, a_k, a_{-k}, y)$, we also have $\gamma_k z_k \in Z_k(x, \bar{a}_k, a_{-k}, y)$. Thus, $\gamma_k \in \Gamma_k(x, \bar{a}_k, a_{-k}, y, z_k)$. Hence, $\bar{\gamma}_k \leq \gamma_k$.
- E_k is non-decreasing in a_{-k}: Let $\bar{a}_{-k} \geq a_{-k}$, $E_k(x, a_k, a_{-k}, y, z_k) \equiv \gamma_2$, and $E_k(x, a_k, \bar{a}_{-k}, y, z_k) \equiv \bar{\gamma}_2$. Define $l \equiv -k$ so that $-l = k$. Then $\bar{a}_l \geq a_l$, $E_k(x, a_k, a_{-k}, y, z_k) = E_{-l}(x, a_l, a_{-l}, y, z_{-l}) \equiv \gamma_2$, and $E_k(x, a_k, \bar{a}_{-k}, y, z_k) = E_{-l}(x, a_{-l}, \bar{a}_l, y, z_{-l}) \equiv \bar{\gamma}_2$. Since, we are operating in the domain where $E_k \equiv E_{-l}$ is well-defined, we have $P(x, a_l, a_{-l}, y) \neq \emptyset$ and $P(x, \bar{a}_l, a_{-l}, y) \neq \emptyset$. Hence, Assumption (ESCU) implies that $Z_{-l}(x, \bar{a}_l, a_{-l}, y) \subset Z_{-l}(x, a_l, a_{-l}, y)$. Hence, $Z_k(x, a_k, \bar{a}_{-k}, y) \subset Z_k(x, a_k, a_{-k}, y)$. Hence, since $\bar{\gamma}_k z_k \in Z_k(x, a_k, \bar{a}_{-k}, y)$, we also have $\bar{\gamma}_k z_k \in Z_k(x, a_k, a_{-k}, y)$. Thus, $\bar{\gamma}_k \in \Gamma_k(x, a_k, a_{-k}, y, z_k)$. Hence, $\gamma_k \leq \bar{\gamma}_k$.
- E_k is constant in y_o and x_o: This follows because Assumption (INDo) implies that $P(x_z, x_o, a, y_z, y_o) = P(x_z, \bar{x}_o, a, y_z, \bar{y}_o)$ for all $\langle x_z, x_o, a, y_z, y_o \rangle$ and $\langle x_z, \bar{x}_o, a, y_z, \bar{y}_o \rangle$ in

$\boldsymbol{\Theta}$. Hence, $\Gamma_k(x_z, x_o, a, y_z, y_o, z_k) = \Gamma_k(x_z, \bar{x}_o, a, y_z, \bar{y}_o, z_k)$. Hence, $E(x_z, x_o, a, y_z, y_o, z_k) = E(x_z, \bar{x}_o, a, y_z, \bar{y}_o, z_k)$.

• E_k is jointly continuous in its arguments: Proof is similar to proof of joint continuity of D_1 in Theorem (IP). ■

Proof of Theorem (BP-ESCU-REPR):

$\Longrightarrow$ of (7.3): Follows directly from Parts (ii) of Theorems (IP) and (EG-ESCU).

$\Longleftarrow$ of (7.3): We show that $\langle x, a, y, z\rangle \notin \mathcal{T}$ implies $\langle x, a, y, z\rangle \notin \mathcal{T}_1 \cap T_2$, *i.e.,* $\langle x, a, y, z\rangle \notin \mathcal{T}_1$ or $\langle x, a, y, z\rangle \notin T_2$.

Case 1. Suppose $\langle x, a, y, z\rangle \notin \mathcal{T}$ and $\langle x, a, y, z\rangle \in T_2$. We show that $\langle x, a, y, z\rangle \notin \mathcal{T}_1$. $\langle x, a, y, z\rangle \in T_2$ implies $\gamma_1 \equiv E_1(x, a, y, z_1) \leq 1$ and $\gamma_2 \equiv E_2(x, a, y, z_2) \leq 1$. Hence, $\gamma_1 z_1 \in Z_1(x, a, y)$ and $\gamma_2 z_2 \in Z_1(x, a, y)$. Hence, $\langle \gamma_1 z_1, \gamma_2 z_2\rangle \in Z_1(x, a, y) \times Z_2(x, a, y)$. Assumption (EG-IND$z$) implies $\langle \gamma_1 z_1, \gamma_2 z_2\rangle \in P(x, a, y)$. Define $\gamma = \max\{\gamma_1, \gamma_2\}$. Then $\gamma \leq 1$, $\gamma z_1 \geq \gamma_1 z_1$, and $\gamma z_2 \geq \gamma_2 z_2$. Hence (iii) of Remark 2 implies $\langle \gamma z_1, \gamma z_2\rangle \in P(x, a, y)$. Define $\lambda_1 = D_1(x, a, y, z)$. Suppose $\lambda_1 = 1$. Then, by definition of D_1, we have $\langle y, z\rangle \in \mathcal{P}(x, a)$ or $\langle x, a, y, z\rangle \in \mathcal{T}$, which is a contradiction to the maintained assumptions of this case. Hence, $\lambda_1 \neq 1$. Suppose $\lambda_1 < 1$. Then, by definition of D_1, we have $\langle \frac{y}{\lambda_1}, \frac{z}{\lambda_1}\rangle \in \mathcal{P}(x, a)$. Note that $y < \frac{y}{\lambda_1}$. Hence, Assumptions (FDo) and (FDy_z) imply that $\langle y, \frac{z}{\lambda_1}\rangle \in \mathcal{P}(x, a)$. Hence, $\frac{z}{\lambda_1} \in \mathcal{P}(x, a, y) \subseteq P(x, a, y)$ (by (i) of Remark 2). We show that $z \notin P(x, a, y)$. Suppose not, *i.e.,* suppose $z \in P(x, a, y)$. Under the maintained assumptions $z \notin \mathcal{P}(x, a, y)$. Assumption ($R2$) implies $\kappa z \notin \mathcal{P}(x, a, y)$ for all $\kappa > 1$. But this contradicts $\frac{z}{\lambda_1} \in \mathcal{P}(x, a, y)$. Hence, $z \notin P(x, a, y)$. But $\gamma z \in P(x, a, y)$, $\frac{z}{\lambda_1} \in P(x, a, y)$, and $\gamma z < z < \frac{z}{\lambda_1}$. Hence, z is a convex combination of γz and $\frac{z}{\lambda_1}$ and, since the set $Z_k(x, a, y)$ is a closed and convex interval, we have $z \in P(x, a, y)$. This is a contradiction to $z \notin P(x, a, y)$. Hence, it is not the case that $\lambda_1 < 1$. Hence, $\lambda_1 = D_1(x, a, y, z) > 1$. Hence, by definition of $\mathcal{T}_1$, $\langle x, a, y, z\rangle \notin \mathcal{T}_1$.

Case 2. Suppose $\langle x, a, y, z\rangle \notin \mathcal{T}$ and $\langle x, a, y, z\rangle \in \mathcal{T}_1$. We show that $\langle x, a, y, z\rangle \notin T_2$. $\langle x, a, y, z\rangle \in \mathcal{T}_1$ implies $\lambda_1 \equiv D_1(x, a, y, z) \leq 1$. Hence, by definition of D_1, we have $\langle \frac{y}{\lambda_1}, \frac{z}{\lambda_1}\rangle \in \mathcal{P}(x, a)$. Hence, $\frac{z}{\lambda_1} \in \mathcal{P}(x, a, \frac{y}{\lambda_1}) \subseteq P(x, a, \frac{y}{\lambda_1})$ (by (i) of Remark 2). Note $\lambda_1 \neq 1$, for otherwise, by definition of D_1, we have $z \in \mathcal{P}(x, a, y)$. Hence, $\langle x, a, y, z\rangle \in \mathcal{T}$, which is a contradiction to the maintained assumptions. Hence, $\lambda_1 < 1$. Since $y < \frac{y}{\lambda_1}$, Assumptions (FDo) and (FDy_z) imply $\frac{z}{\lambda_1} \in \mathcal{P}(x, a, y) \subseteq P(x, a, y)$. Define $\gamma_1 \equiv$

$E_1(x,a,y,z_1)$ and $\gamma_2 \equiv E_2(x,a,y,z_2)$. Then $\gamma_1 z_1 \in Z_1(x,a,y)$ and $\gamma_2 z_2 \in Z_2(x,a,y)$. Hence, $\langle \gamma_1 z_1, \gamma_2 z_2\rangle \in Z_1(x,a,y) \times Z_2(x,a,y) = P(x,a,y)$ (under Assumption (EG-INDz)). Define $\gamma \equiv \max\{\gamma_1,\gamma_2\}$. Then $\gamma z_1 \geq \gamma_1 z_1$ and $\gamma z_2 \geq \gamma_2 z_2$. Hence, (iii) of Remark 2 implies $\langle \gamma z_1, \gamma z_2\rangle \in P(x,a,y)$. Two cases are possible: $\gamma \leq 1$ (which implies $\gamma_1 \leq 1$ and $\gamma_2 \leq 1$) and $\gamma > 1$ (which implies $\gamma_1 > 1$ or $\gamma_2 > 1$). Suppose $\gamma \leq 1$. First we show that $\gamma \neq 1$. Suppose not, *i.e.,* suppose $\gamma = 1$. Then, $z \in P(x,a,y)$. Since $\langle x,a,y,z\rangle \notin \mathcal{T}$, we have $z \notin \mathcal{P}(x,a,y)$. Assumption ($R2$) implies $\theta z \notin \mathcal{P}(x,a,y)$ for all $\theta > 1$. That means $\frac{z}{\lambda_1} \notin \mathcal{P}(x,a,y)$. Which is a contradiction. Hence, $\gamma \neq 1$, and $\gamma \leq 1$ implies $\gamma < 1$. The above argument also implies that $z \notin P(x,a,y)$.[38] Hence, we have $\gamma z < z < \frac{z}{\lambda_1}$, with $\gamma z \in P(x,a,y)$ and $\frac{z}{\lambda_1} \in P(x,a,y)$. Hence, since z is a convex combination of γz and $\frac{z}{\lambda_1}$ and the set $Z_k(x,a,y)$ is a closed and convex interval, we have $z \in P(x,a,y)$. Which is a contradiction. Hence, it is not the case that $\gamma \leq 1$. Hence $\gamma > 1$. Hence, $\gamma_1 > 1$ or $\gamma_2 > 1$. Hence, $E_1(x,a,y,z_1) > 1$ or $E_2(x,a,y,z_2) > 1$. Hence, $\langle x,a,y,z\rangle \notin T_2$. ■

Proof of Theorem (BP-ESCU-EFF):

Suppose $\langle y,z\rangle$ is a strictly efficient point of $\mathcal{P}(x,a)$ but $D_1(x,a,y,z) \neq 1$ or $E_1(x,a,y,z_1) \neq 1$ or $E_2(x,a,y,z_2) \neq 1$. Since $\langle y,z\rangle$ is a strictly efficient point of $\mathcal{P}(x,a)$, we have $\langle y,z\rangle \in \mathcal{P}(x,a)$. Hence, Theorem (BP-REPR) implies $D_1(x,a,y,z) \leq 1$ and $E_k(x,a,y,z_k) \leq 1$ for $k = 1,2$.

<u>Case 1:</u> Suppose $\lambda_1 \equiv D_1(x,a,y,z) < 1$.

Then $\langle \frac{y}{\lambda_1}, \frac{z}{\lambda_1}\rangle \in \mathcal{P}(x,a)$. $\frac{y}{\lambda_1} > y$ and $\frac{z}{\lambda_1} > z$. If Assumption (INDz) is true, then $\mathcal{P}(x,a,z) = \mathcal{P}(x,a,\frac{z}{\lambda_1})$. If Assumption (DET$z$) is true, then $\mathcal{P}(x,a,\frac{z}{\lambda_1}) \subset \mathcal{P}(x,a,z)$. Hence, in either case, $\langle \frac{y}{\lambda_1}, z\rangle \in \mathcal{P}(x,a)$. Hence, $\langle y,z\rangle$ is not a strictly efficient point of $\mathcal{P}(x,a)$, which is a contradiction.

<u>Case 2:</u> Suppose $\gamma_k \equiv E_k(x,a,y,z_k) < 1$ for some $k = 1,2$.

Definition of E_k and Assumption (EG-INDz) imply $\langle \gamma_1 z_1, \gamma_2 z_2\rangle \in P(x,a,y)$. We also have $z \in \mathcal{P}(x,a,y) \subseteq P(x,a,y)$ (from (i) of Remark 2). Hence, $\gamma_k z_k \in P(x,a,y,\gamma_{-k}z_{-k})$ and $z_k \in P(x,a,y,z_{-k})$. Assumption (EG-INDz) implies $P(x,a,y,\gamma_{-k}z_{-k}) = P(x,a,y,z_{-k})$. Hence, $\gamma_k z_k \in P(x,a,y,z_{-k})$. We show that $\gamma_k z_k \in \mathcal{P}(x,a,y,z_{-k})$. Suppose not, *i.e.,* $\gamma_k z_k \notin \mathcal{P}(x,a,y,z_{-k})$. Then, since $\gamma_k z_k \in P(x,a,y,z_{-k})$, Assumption ($R2$) implies,

[38] For if $z \in P(x,a,y)$ then, since $z \notin \mathcal{P}(x,a,y)$, Assumption ($R2$) implies $\frac{z}{\lambda_1} \notin \mathcal{P}(x,a,y)$ which is a contradiction.

$\theta z_k \notin \mathcal{P}(x, a, y, z_{-k})$ for all $\theta > \gamma_k$. In particular, since $\gamma_k < 1$, this means that $z_k \notin \mathcal{P}(x, a, y, z_{-k})$, which is a contradiction. Hence, $\gamma_k z_k \in \mathcal{P}(x, a, y, z_{-k})$. Hence, $\langle z, y \rangle$ is not a strictly efficient point of $\mathcal{P}(x, a)$, which is a contradiction. ■

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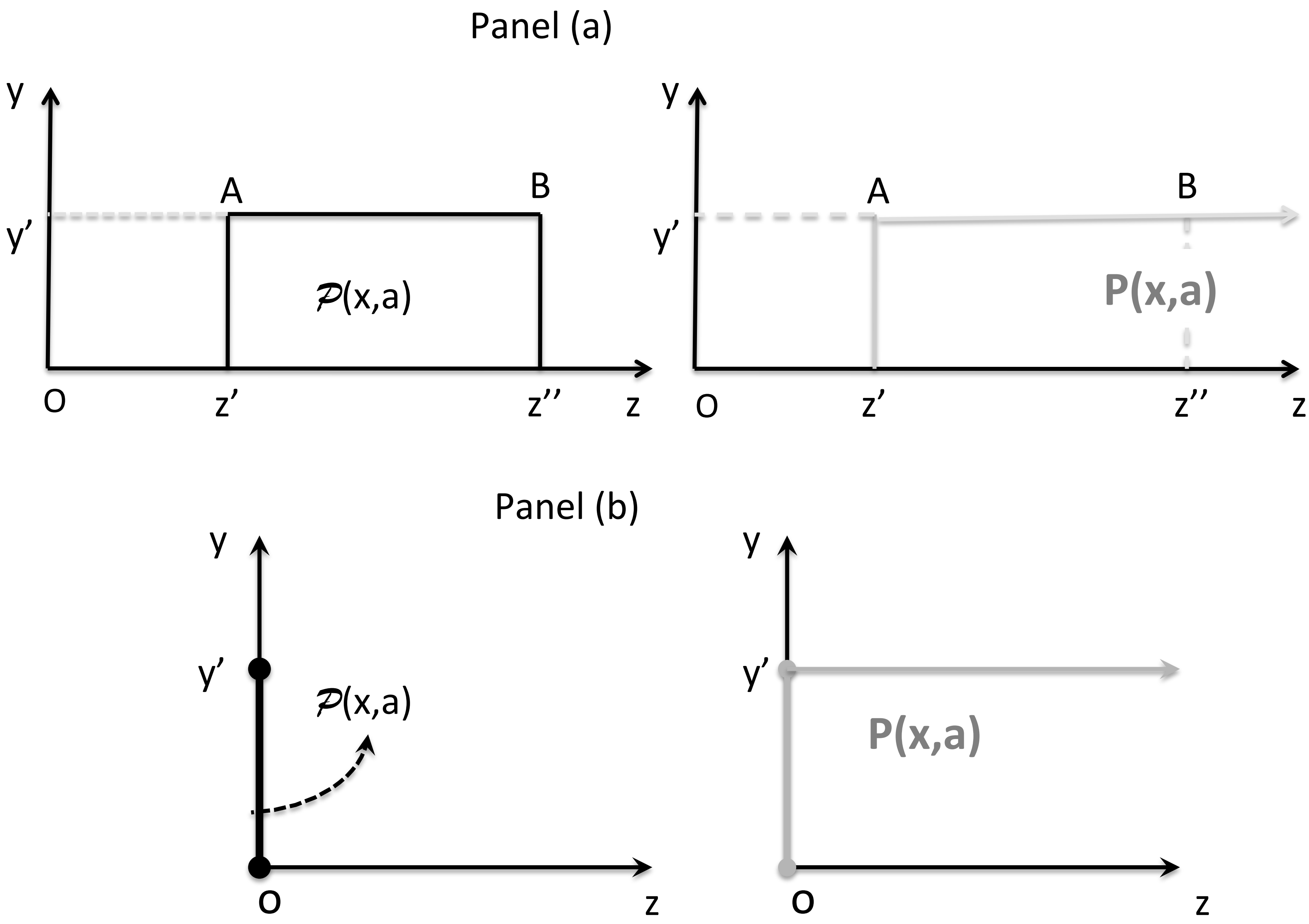
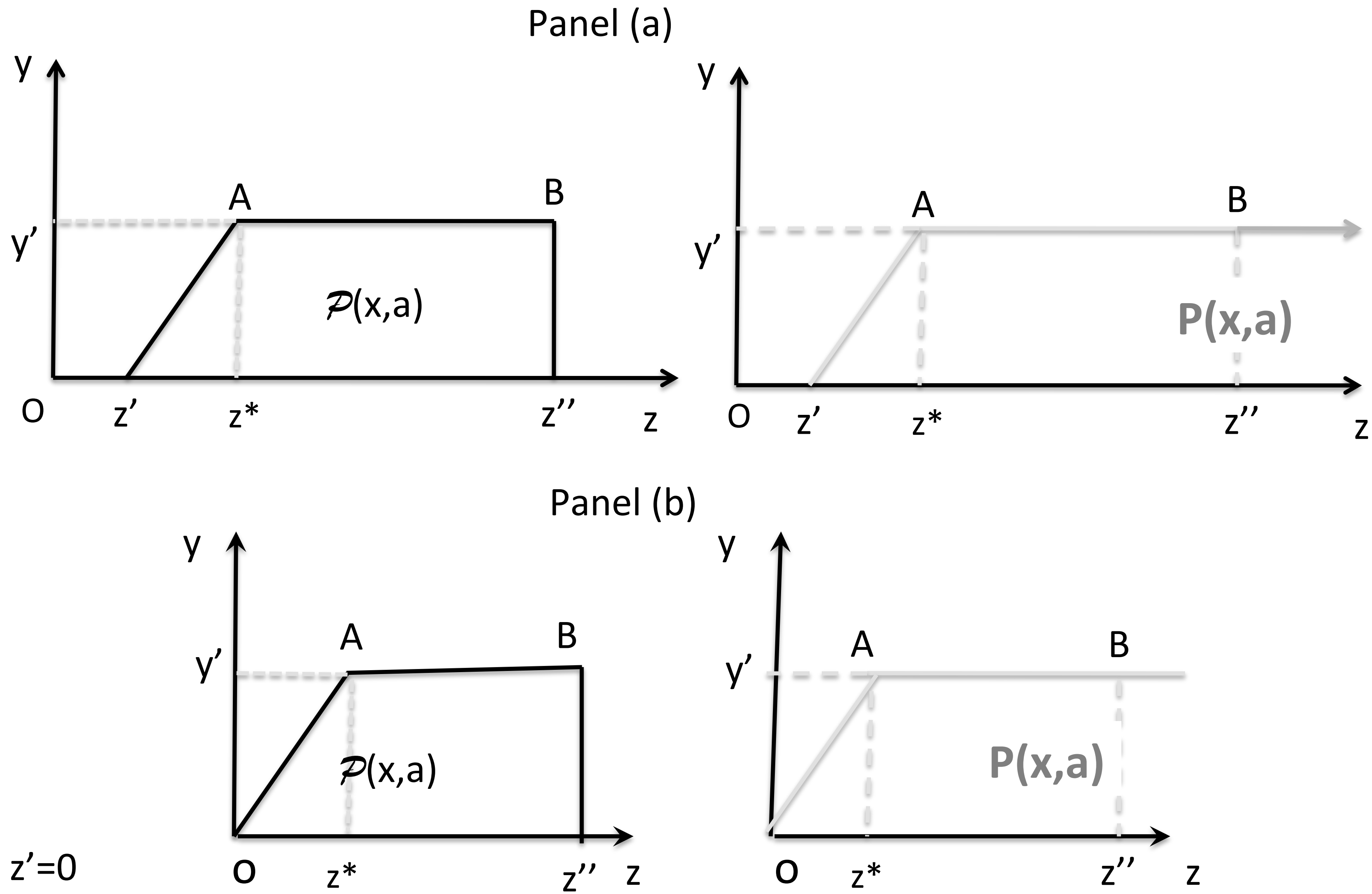


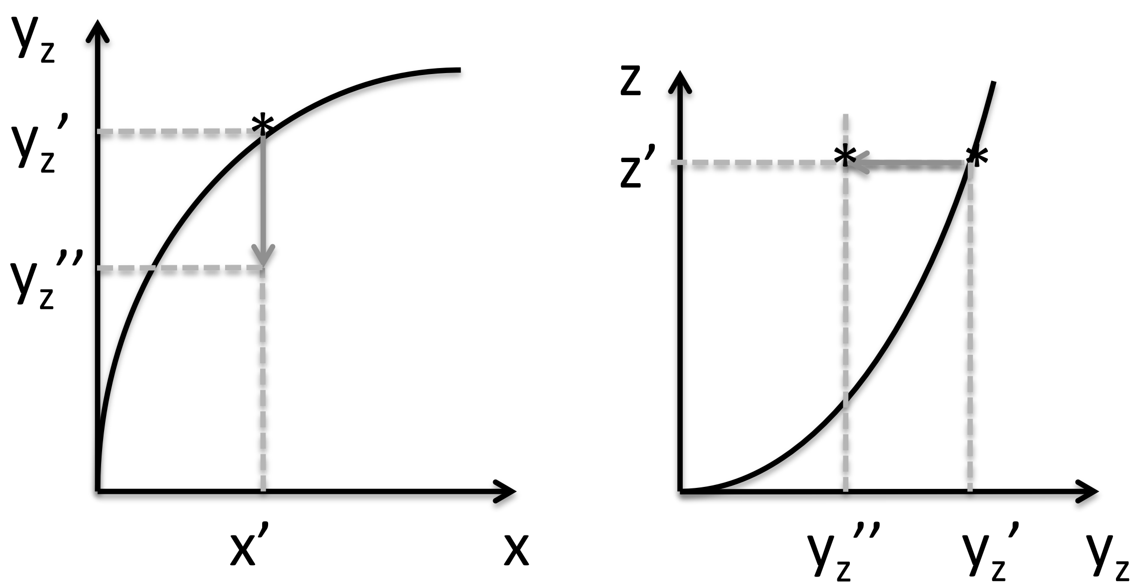
$m_z=0,\ x_z=0,\ x_o \neq 0.$

Figure 1



$m_z \neq 0$, $x_z = 0$, $x_o \neq 0$.

Figure 2



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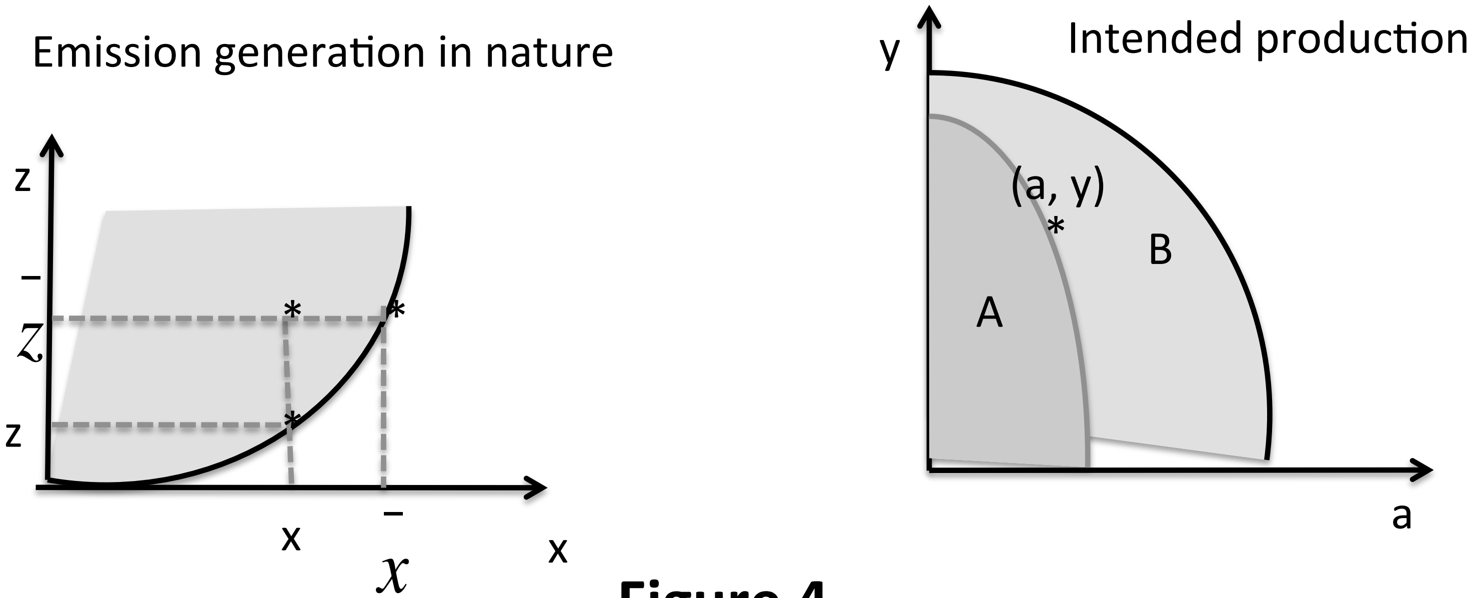

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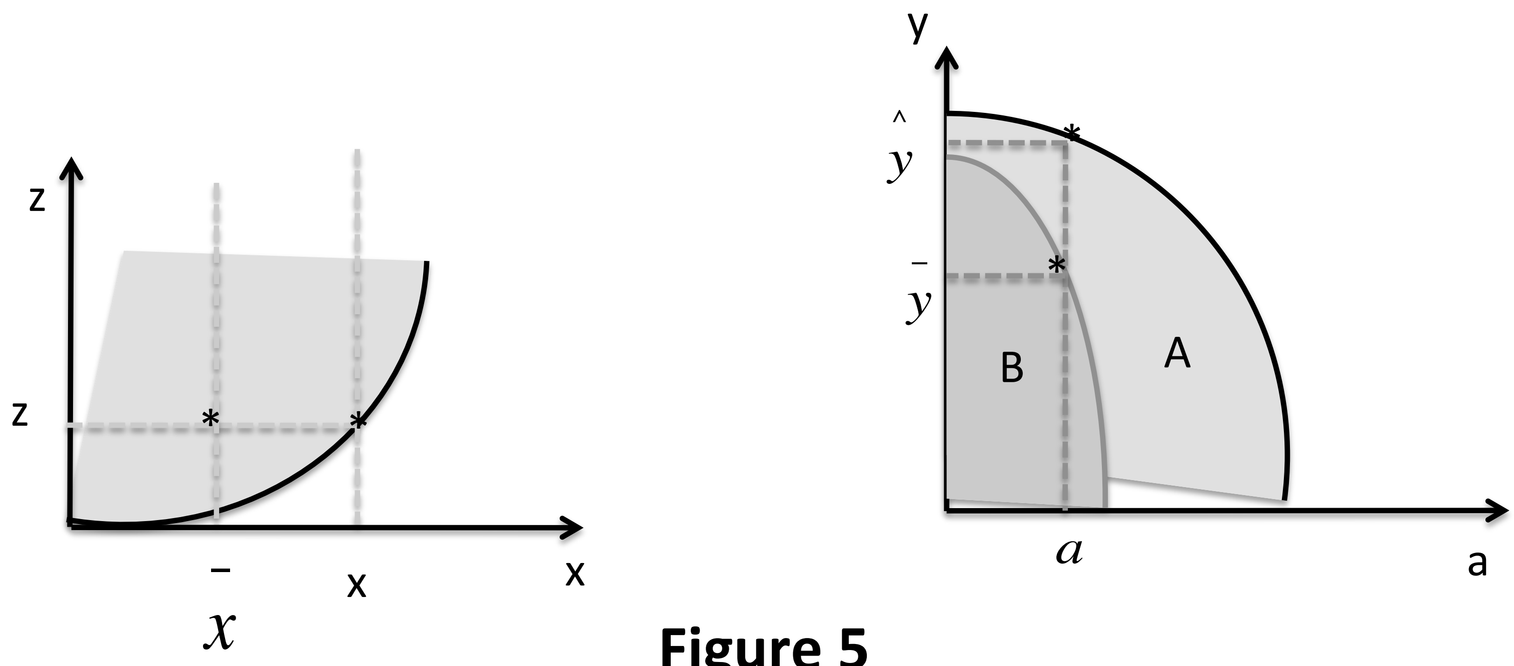

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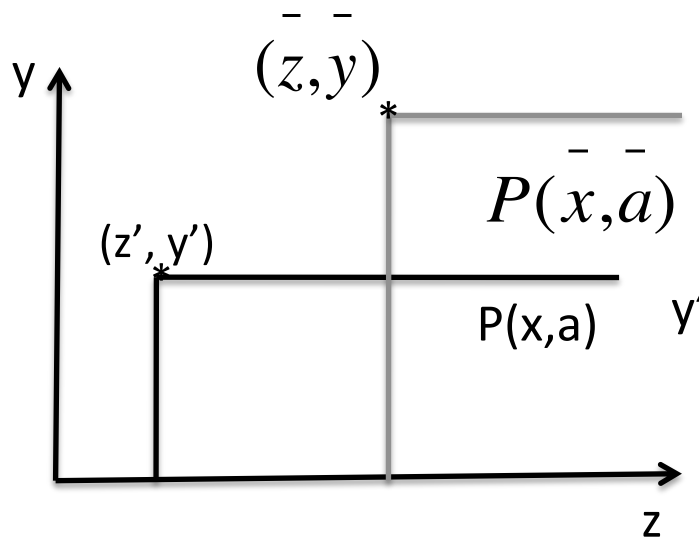

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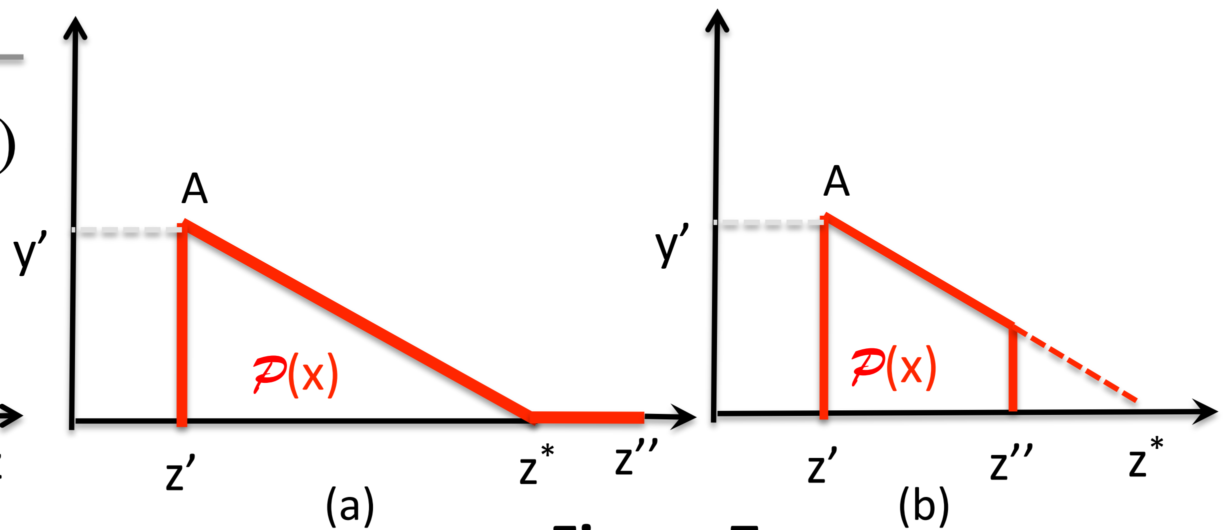

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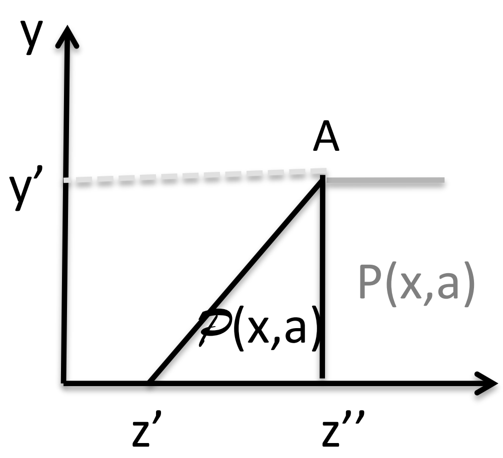

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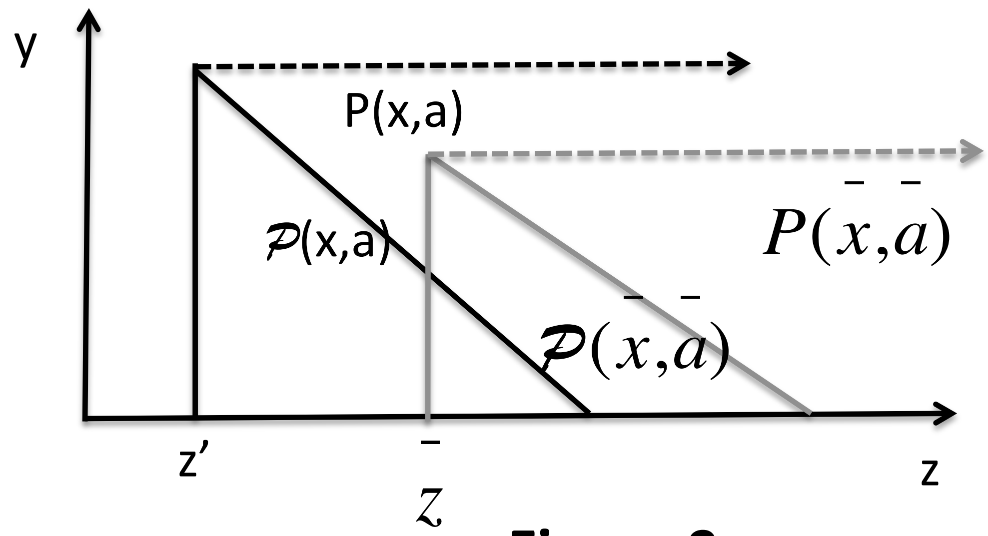

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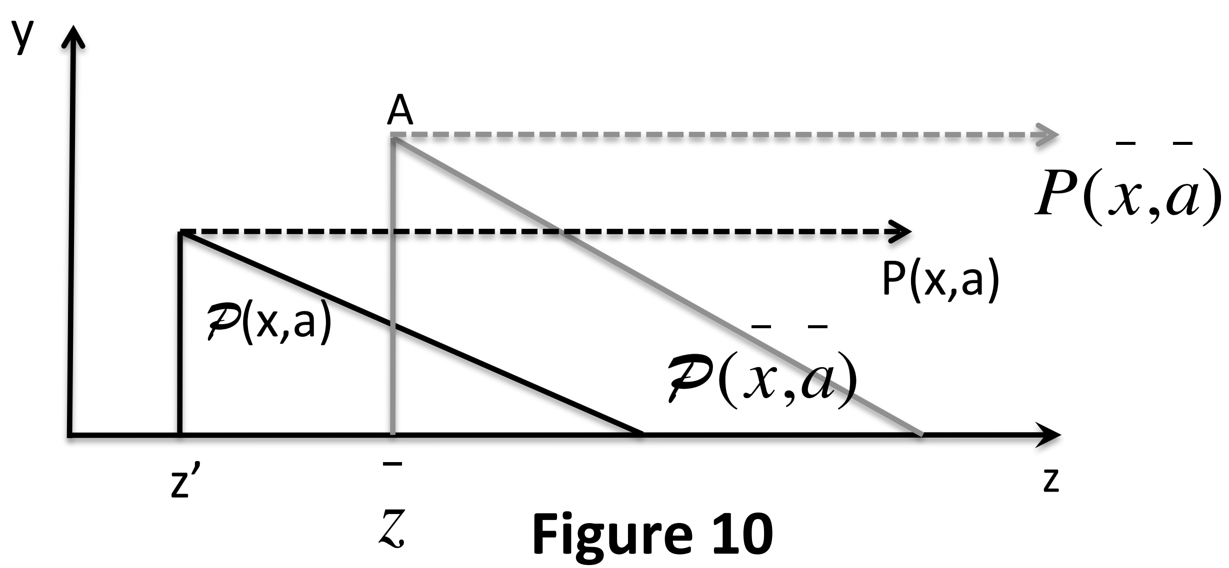

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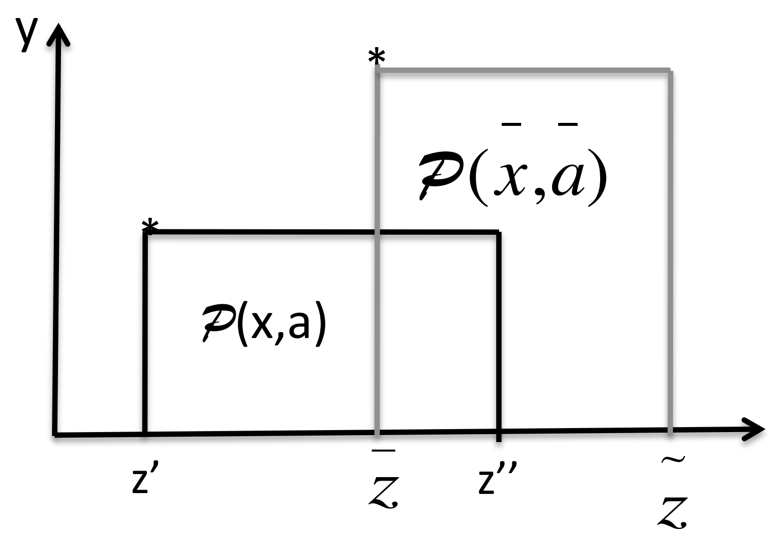

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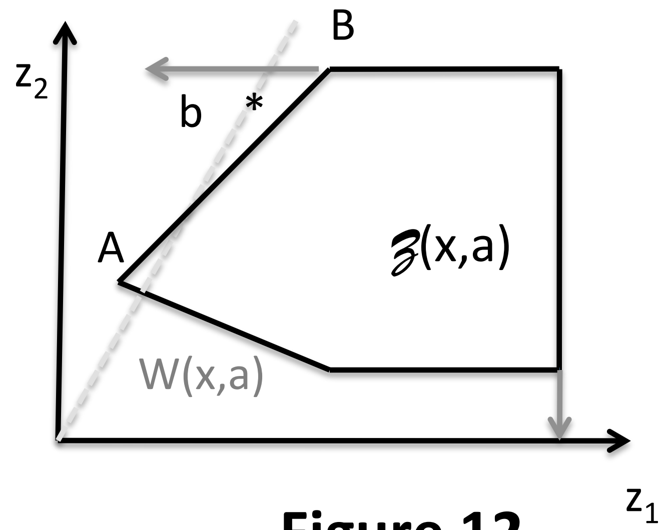

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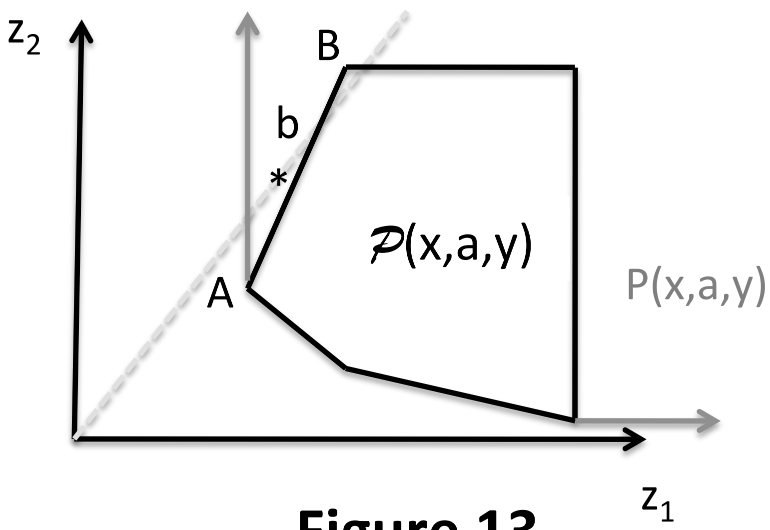

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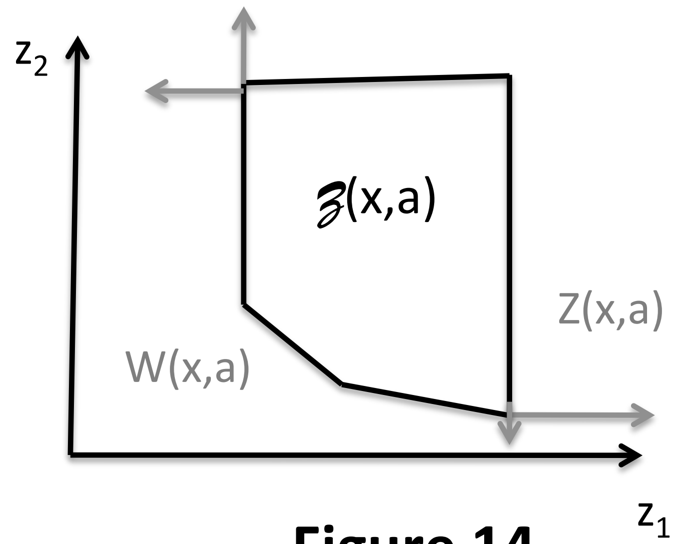

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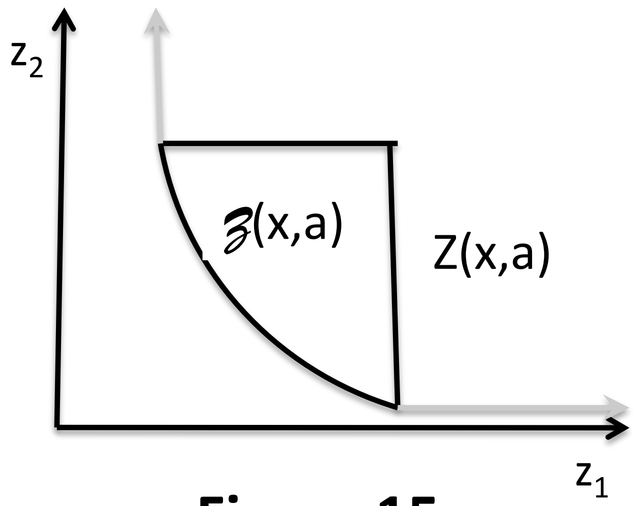

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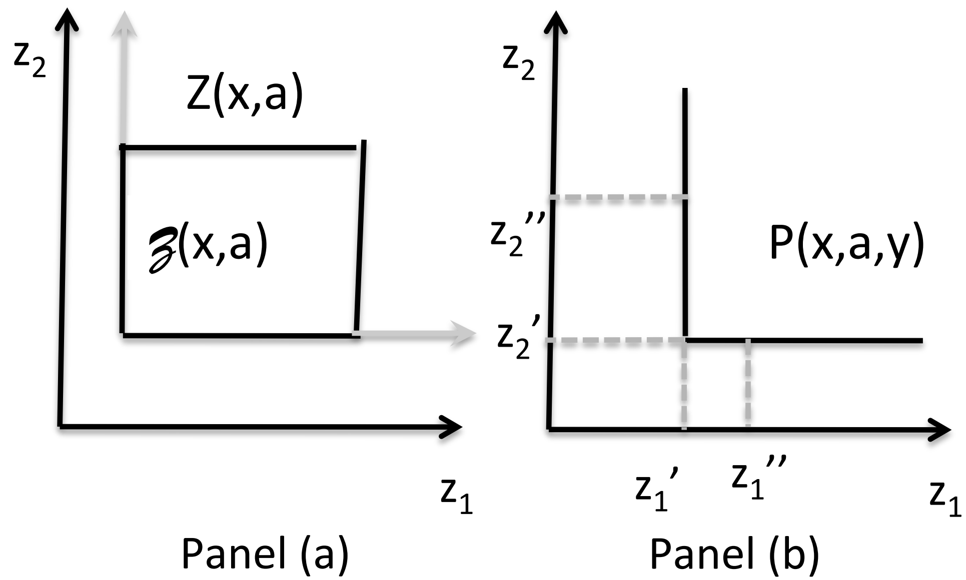

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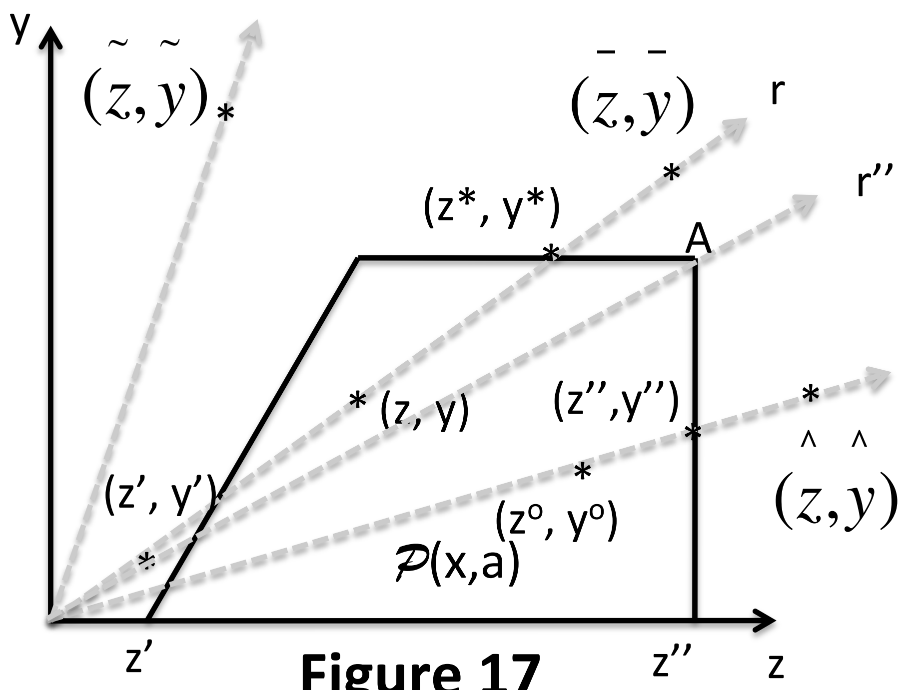

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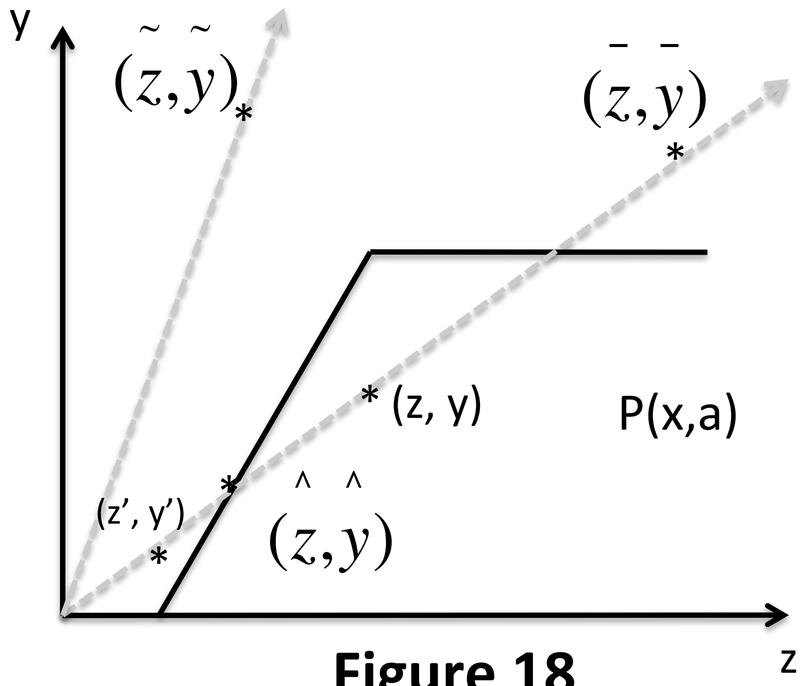

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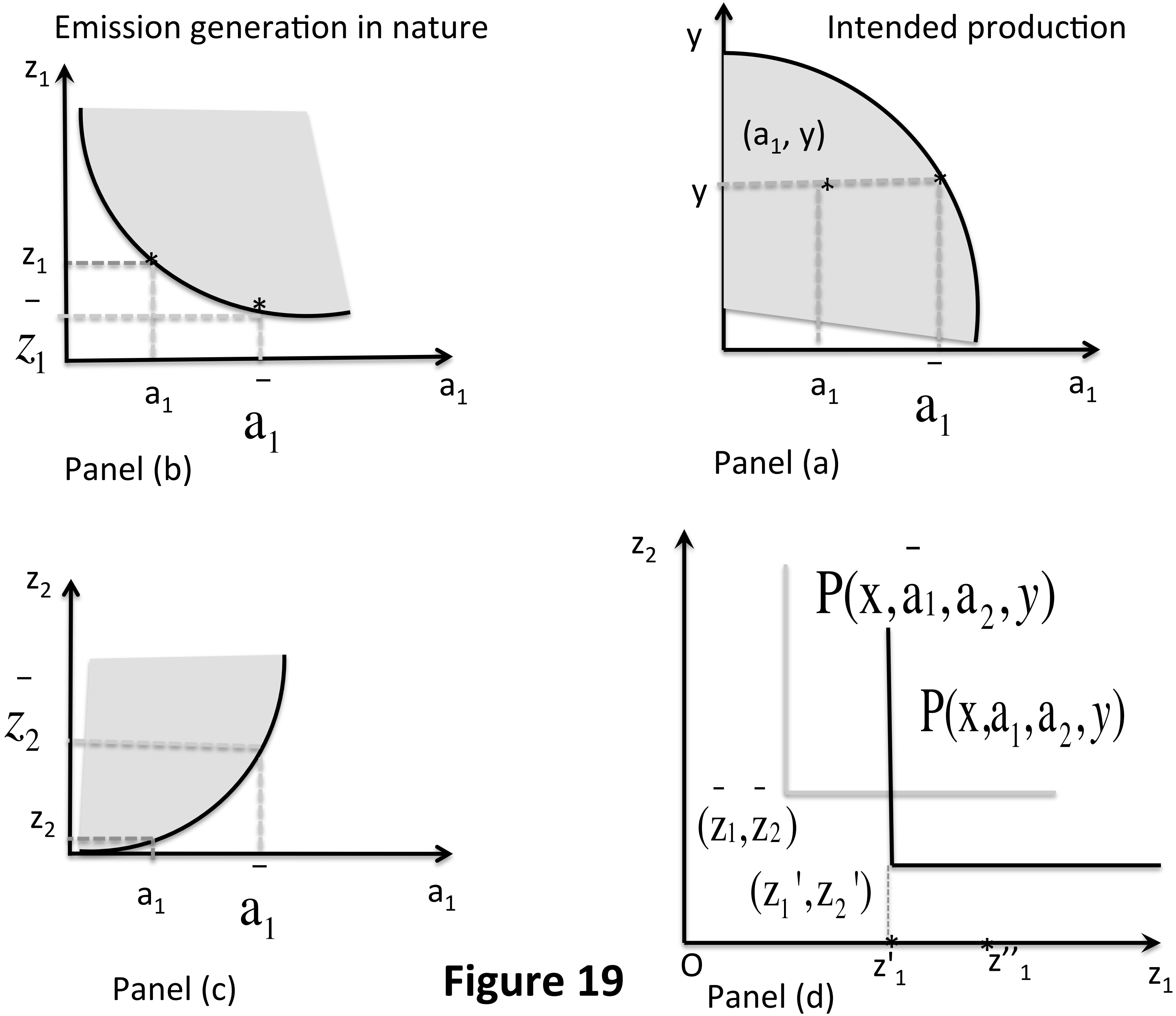

Emission generation in nature
Intended production
Panel (b)
Panel (a)
Panel (c)
Panel (d)
(a_1, y)
$P(x, \bar{a}_1, a_2, y)$
$P(x, a_1, a_2, y)$
$(\bar{z}_1, \bar{z}_2)$
(z_1', z_2')
z'_1
z''_1
O

Figure 19