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The Theory of By-Production of Emissions and Capital-Constrained Non-Cooperative Nash Outcomes of a Global Economy.

Sushama Murty

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Sushama Murty*

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*Department of Economics, University of Exeter Business School, University of Exeter:

s.murty@exeter.ac.uk

Abstract

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The reduced form approaches that are commonly adopted in the literature to model emission-generating technologies (EGTs) do not distinguish between emission-causing and non-emission causing goods in production. We provide a new set of axioms to describe EGTs. Technologies that satisfy these axioms are called by-production technologies (BPTs). A distance function representation of BPTs is derived and it is shown that a BPT can be decomposed into a standard neo-classical intended-production technology and nature's emission-generation set (the relationship in nature between emissions and emission-causing goods). As an illustrative application of the BP approach, we study cross-country differences in emission levels due to cross-country differences in capital endowments at a non-cooperative Nash equilibrium, where emissions impose both local and global externalities. The change in emission levels as we move from capital-poor to capital-rich countries is decomposed into income and substitution effects. The latter are a result of changes in the trade-off between intended-production and emission-generation, which is attributed to diminishing returns to emission-causing inputs or cleaning-up activities, while the nature of the former is governed by the assumption that emission is an inferior good. The implications of increasing returns to capital, substitutability or complementarity between capital and emission-causing inputs such as fuels, extraction costs of fuels, and inter-fuel substitution in production are studied and a set of conditions that result in an environmental Kuznets curve is derived. *JEL classification codes:* Q50, Q56, Q51, O10, O12, D20, D62, D11. *Keywords:* distance function representation of multi-output technology, emission-generating technologies, free and costly disposability, environmental Kuznets curve, environmental externalities, non-cooperative Nash equilibrium, income and substitution effects, inferior good, returns to scale, inter-fuel substitution.

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1. Introduction.

A model of an emission-generating technology (EGT) should distinguish between production of intended outputs by firms and the generation of emissions, which are the unintended outputs. In nature, emissions are caused by certain goods under certain physical or chemical conditions. These emission-causing goods could form a part of the intended production activities of a firm either as inputs[1] or as intended outputs.[2] Intended production by firms, hence, triggers the physical and chemical reactions that are conducive for emission generation in nature, and generates emissions as *by-products.* Murty, Russell, and Levkoff (MRL) [2011] have called this simultaneity in the processes of intended production by firms and emission generation by nature that arises because of the use or production of emission-causing goods by firms as *by-production* (BP). Abatement (or reduction) of emissions involves either (i) reducing the scale of intended production, which reduces the use or production of goods that cause emissions in nature or (ii) diverting a part of the firm's resources (inputs) into explicit production of *cleaning-up activities*, such as operating end-of-pipe (effluent) treatment plants, use of scrubbers, afforestation, *etc.*

In contrast, barring a few recent exceptions, much of the theoretical and applied literature concerned with emission generation takes a reduced form approach to model the relationship between intended outputs of firms and emission generation.[3]

[1] *E.g.,* coal.

[2] *E.g.,* certain varieties of cheese that liberate strong odours as emissions.

[3] Exceptions include MRL, Førsund [2009], and Coelli, Lauwers, and Van Huylenbroeck [2007]. Some of these papers are motivated by the work of Nobel laureate Ragnar Frisch [1965], who showed that there are many instances of production when more than one production relation may be required to model a production technology, while others are motivated by the the material balance approach of Ayres and Kneese [1969].

The first is a partial-equilibrium welfare approach, where the reduced form relation between emissions and intended outputs of firms excludes explicit or implicit mention of goods that cause emissions in nature. In many such papers, the basic idea is that there is an initial endowment of a numeraire commodity which can potentially be allocated between an aggregate measure of abatement activities and consumption. The more the numeraire is allocated to abatement the less is available for good consumption, and hence the positive relation between emission levels and the level of "good" consumption.[4] A motivation for these works is to study, design, and compare welfare improving environmental policies within a partial-equilibrium framework.

The second is an approach motivated by measurement issues such as measuring the consequences of emission generation on technical efficiency, growth, productivity, *etc.,* and assessment of shadow values of emissions. A motivation for these works is to provide inputs to governmental agencies for setting the right levels of policy instruments and targets to regulate emission-generating units. Models of technologies in this approach are described in the space of all inputs, intended outputs, and emissions. Nonetheless, they continue to be reduced form specifications, as they do not attribute emissions to inputs or intended outputs that cause emissions in nature. Rather, the positive relationship between emissions and intended outputs is explained in terms of abatement (in particular, cleaning-up) activities of firms, which are not modeled explicitly. It is recognized in this literature that, unlike in the case of the intended outputs, the technology does not satisfy output free disposability with respect to emissions. To obtain the reduced-form positive relation between emission generation and intended production, emissions are either treated

[4] See, *e.g.,* the text book on environmental economics by Kolstad [2000]. Among other several notable examples is the classic paper by Weitzman [1974].

as inputs (the input approach)[5] or as *weakly* disposable outputs (the output approach).[6]

MRL show that models of EGT in the measurement approach exhibit trade-offs among inputs, intended outputs, and emissions which are intuitively unacceptable.[7] MRL show that a technology obtained as a composition of two technologies: (a) an intended production technology and (b) nature's emission-generation mechanism, generates all the intuitively expected trade-offs between various goods. MRL do not however provide a complete characterization of the composite technology. It is the overall technology, and not its individual components (a) and (b), that we observe empirically. Modeling the overall technology requires substantial a-priori knowledge of its properties.

This brings us to the *first* of the *two* main objectives of this paper: we introduce a new set of axioms for an EGT. Any technology that satisfies these axioms is called a by-production technology (BPT). These axioms include standard output or input free disposability of *all* (including emission-causing) intended outputs and non-emission causing inputs. The emission-causing inputs and cleaning-up activity of the firm on the other hand satisfy more complex disposability assumptions, which we call *conditional* costly disposability. It is shown that this new set of axioms lend themselves to a very convenient functional representation of a BPT. Since we are dealing with technologies producing multiple outputs (intended and unintended), the most convenient functional representation is offered by the use of distance functions.[8] Since this paper introduces a completely new

5 See, *e.g.,* Baumol and Oates [1988], Cropper and Oates [1992], Reinhard, Lovell, and Thijhssen [1999], and Reinhard, Lovell, and Geert [2000].

6 See, *e.g.,* Färe, Grosskopf, Noh, and Yaisawarng [1993], Coggins and Swinton [1994], Hailu and Veeman [1999], Murty and Kumar [2002, 2003], Färe, Grosskopf, Lovell, and Pasurka [1989], Färe, Grosskopf, Noh, and Weber [2005], and Boyd and McClelland [1999]. See Zhou and Poh [2008] for a comprehensive survey of over a hundred papers employing this approach.

7 See also Murty [2010b] for an example that demonstrates this, *e.g.,* it is possible in these models to observe a negative relation between emissions and an emission-causing input such as coal. This is precisely because these works do not model the link in nature between emissions and the emission-causing goods used or produced in intended production.

8 Also called gauge functions or transformation functions in the literature, the use of distance functions is common in production theory since their introduction by Malmquist [1953] and Shephard [1953]. They have been extensively studied in the works of Gorman [1970], Blackorby, Russell, and Primont [1978],

set of axioms to characterize EGTs, the functional representations obtained here are novel and quite different from those in the existing literature.[9] In particular, two distance functions are derived from an arbitrarily given BPT, one capturing the upper bounds on intended outputs and the other capturing the lower bounds on emission generation in nature. It is shown that a production vector belongs to a BPT if and only if *both* these distance functions, evaluated at this production point, take a value less than or equal to one. Properties of these two distance functions are carefully derived and it is shown that, by employing these two implicit functions, one can decompose the associated BPT into its underlying intended production technology and the nature's emission generation set. Further, intersection of technologies represented by any arbitrary pair of implicit functions possessing the same properties as the new distance functions that we have defined, results in a BPT. Thus, our new set of axioms provide both necessary and sufficient conditions for decomposition of an EGT into its underlying intended production and emission generation components. This forms the agenda of Sections 2 to 5. The model of technology that we consider in these sections is very general: it allows for emissions of a producing unit to impose beneficial or detrimental external effects on its own intended production, and it allows for cases where there is a *jointness* in the production of multiple emissions.

The *second* objective of this paper is to demonstrate that the BP approach to modeling EGT leads to far richer sets of results and explanations to empirically observed economic phenomenon than those derived from the partial-equilibrium welfare approach. The more elaborate specification of an EGT in the BP approach enables one to move towards a general equilibrium framework, where many pertinent economic laws can be conveniently

Fuss and McFadden [1978], Färe and Primont [1995], *etc.,* and applied in many areas of economics such as index number theory (construction of technical efficiency, productivity, and cost of living indexes) and the theory of optimal taxation in public economics (see Deaton [1979]). For an excellent discussion and survey of the concept and its applications see Russell [1998].

[9] In the existing literature, distance function representations are primarily based on the assumptions of weak disposability and null jointness employed in the output approach to emissions.

modeled into the problem. We hope to demonstrate the richness of the results that can be derived from the BP approach through one application of this approach.

There is a large empirical and partial-equilibrium based theoretical literature that seeks to measure and explain cross-country differences in emission generation. Primarily, these differences are associated with the cross-country differences in the level of economic development. Guesnerie [2008] explains differences in emission choices of countries in terms of differences in preferences: globally, over the entire consumption set, developed countries value environment more than less developed countries. The environmental Kuznets curve phenomenon, which depicts an inverted U shape relationship between an indicator of economic development and various emissions has been tested by many applied papers.[10] A large theoretical literature seeks to find restrictions on the technologies and preferences that are consistent with this phenomenon. Under the restrictions imposed by these models, cross-country differences in the endowment of a numeraire commodity may imply technological or institutional bottlenecks for poorly endowed countries, resulting in higher opportunity costs of abatement in these countries, to which is attributed their lower marginal willingness to pay (MWTP) for emission reductions and higher emission levels.[11] The reduced form specification of technology in these models precludes any link between emissions and emission-causing goods seen in nature. Many interesting features regarding the process of emission reduction as we move from poor to rich countries are hence missed out. In contrast, our BP approach allows us to model all aspects of neoclassical production theory, such as the economic law of diminishing returns, as well as nature's emission-generating mechanism. It also allows us to model a rich set of production features which are realistic and pertinent to the question at hand, such as extraction costs of emission-causing (fuel) inputs, inter-fuel differences in emission intensity, the possibility of

[10] See, *e.g.,* Grossman and Krueger [1995] and Mason and Swanson [2001] and the references therein.

[11] See, *e.g.,* Stokey [1998], Jones and Manuelli [2001], Andreoni and Levinson [2001], and Israel and Levinson [2004].

complementarity between the use of capital equipment and fuel, and the possibility that there could be non-decreasing returns to capital. The most general of such models could however be quite intractable for analysis. However, there is a minimalistic model, where each of these features can be minimally represented to form a part of a tractable analysis. This is the approach we adopt in this paper. In our simple illustrative application in Section 6, we study a non-cooperative Nash equilibrium of a static global economy in autarky where an emission imposes both a local and a global externality and where countries are identical in all respects (technologies, preferences, and labor endowments) but one: their endowment of a resource that we call *capital*, which is an important input into production. The aim is to study to what extent the sheer differences in the availability of a key productive factor can help explain cross-country differences in emissions. Combined with the assumption that emission is an inferior good, the BP approach to this problem yields the following conclusions at a non-cooperative Nash equilibrium of the global economy:

(1) For any country, the consumption trade-off (or the marginal rate of substitution (MRS)) between intended output and emission is equated to the trade-offs in production between the intended output and emission.

(2) For any country, the trade-offs in production between intended-production and emission generation (or the opportunity costs of reducing emissions) are equalized across all abatement strategies, (*e.g.,* trade-offs resulting from changes in the level of cleaning-up activity or changes in the level of fuel usage, are all equal).

(3) As we move from capital-poor to capital-rich countries, the changes in levels of emission and the intended output can be decomposed into income and substitution effects. The substitution effects arise because the MRS can change as the capital level changes. The change in the MRS is attributed to diminishing returns to fuel inputs *or* cleaning-up activity. Substitution effects vanish if there are no such diminishing returns. Income effects arise because the increase in capital can result in an increase in real income, which is given by the *net marginal product of capital* – the contribution of the additional capital input to intended production *net* of the extraction cost of fuel needed to run the additional capital.

(4) If the use of capital is *not* fuel intensive, *i.e.,* capital and fuel inputs are *not* perfect complements, then as capital increases, it acts as a substitute for fuel inputs in intended production. The use of fuel inputs decreases and diminishing returns implies increased productivity of fuel inputs and hence higher MRS. Substitution effects, hence, involve greater emission and greater intended production. However, as real income also increases with increase in capital and because emission (resp., intended output) is an inferior (resp., normal) good, the income effects imply lower emission and higher intended output production. We show that the income effects dominate the substitution effects: emission (resp., intended output) level decreases (resp., increases) as we move from capital-poor to capital-rich countries. In particular, if there are no diminishing returns to fuel inputs or cleaning-up activity, then the MRS will *not* change as we move from capital-poor to capital-rich countries, though the emission level decreases. Contrast this with the earlier literature, where poor countries are invariably associated with higher MRS (lower MWTP for emission reduction) than the rich countries.

(5) If the use of capital is fuel intensive then, if (i) increasing returns to capital is true, (ii) the net marginal product of capital takes negative values for low levels of capital and positive values for high levels of capital, and (iii) the substitution effects are negligible, then an environmental Kuznets curve phenomenon arises at a non-cooperative Nash equilibrium: there is an inverted U (resp., U) shape relation between capital and emission (resp., intended output) levels. The appearance of an environmental Kuznets curve at a non-cooperative Nash equilibrium in our analysis, implies that there exists a critical level of capital such that countries with capital endowment less than the critical amount cannot reap dividends from increasing returns to capital. Rather, in such countries, a large amount of resources have to be diverted away from intended production towards extraction of fuel needed for running the capital so that it is possible that the net marginal product of capital is negative. Moreover, with no substitution effects, this is true with the MRS being the same across all countries.

(6) Under constant elasticity of inter-fuel substitution, substitution effects depend only on the fuel ratio. The fuel ration is independent of the level of capital and depends on both the marginal extraction costs of the fuels and their relative emission intensities. The ratio of a cleaner to a dirtier fuel is greater than one if the dirtier fuel's cost of extraction is also higher than the cleaner fuel. This ratio is lower the lower is the relative extraction cost of the dirtier fuel. There is a threshold extraction cost of the dirtier fuel below which the equilibrium ratio is lesser than one.

We conclude in Section 7. Several diagrams and examples have been employed throughout the paper to explain our constructs, axioms, and special cases of our results. All proofs have been relegated to the appendix.

2. A model of by-production.

Our by-production model of an EGT has the following components:

- m intended outputs, of which those indexed $1, \ldots, m_z$ (with $m_z \leq m$) cause emissions and the remaining $m_o = m - m_z$ do not. A quantity vector of intended outputs is denoted by $y = \langle y_z, y_o \rangle \in \mathbf{R}^m_+$. Intended outputs are indexed by j, *e.g.,* y_{z_j} is the quantity of the j^{th} emission-causing output.
- n inputs, of which those indexed $1, \ldots, n_z$ (with $n_z \leq n$) cause emissions and the remaining $n_o = n - n_z$ do not. A quantity vector of inputs is denoted by $x = \langle x_z, x_o \rangle \in \mathbf{R}^n_+$. Inputs are indexed by i, *e.g.,* x_{z_i} is the quantity of the i^{th} emission-causing input.

- m' types of emissions. A quantity vector of emissions is denoted by $z \in \mathbf{R}_+^{m'}$. Emissions are indexed by k, *e.g.,* z_k is the quantity of the k^{th} emission.
- A cleaning-up output, whose quantity is denoted by $a \in \mathbf{R}_+$.

An EGT is a set of production vectors of the form $\langle x, a, y, z \rangle = \langle x_z, x_o, a, y_z, y_o, z \rangle \in \mathbf{R}_+^{n+m+m'+1}$ and is denoted by $\mathcal{T} \subset \mathbf{R}_+^{n+m+m'+1}$. Restrictions of $\mathcal{T}$ to various subspaces are denoted by $\mathcal{P}(\cdot)$, *e.g.,* $\mathcal{P}(x, a, z) \subset \mathbf{R}_+^m$ is the restriction of $\mathcal{T}$ to the space of all intended outputs given a fixed quantity vector $\langle x, a, z \rangle$ of all other goods. Similarly, $\mathcal{P}(x, a, y) \subset \mathbf{R}_+^{m'}$ is the restriction of $\mathcal{T}$ to the space of all emissions given a fixed quantity vector $\langle x, a, y \rangle$ of all other goods.

Given a fixed amount of all inputs and cleaning-up activity, there are bounds on both the levels of intended outputs and emissions that are technologically feasible. In particular, it is reasonable to assume that there is an upper bound on intended production and a lower bound on emission generation.[12] The latter reflects the fact that it is *costly* to reduce emissions below such a lower bound. Technical inefficiency may imply that less intended outputs are produced (resp. more emissions are generated) than the levels indicated by the tightest upper bounds (resp. lower bounds) on intended outputs (resp. emissions). One can also envisage upper bounds on emission generation, but if emissions impose detrimental external effects on (other) agents in the economy, then economic efficiency implies that we are more interested in the lower bounds on emission generation. Thus, we derive the *costly-disposal hull* T of the EGT $\mathcal{T}$ as

$$T \equiv \mathcal{T} + \left(\{ 0^{(n+m+1)} \} \times \mathbf{R}_+^{m'} \right).$$

T includes all production vectors in $\mathcal{T}$ as well as production vectors that generate arbitrarily higher levels of emissions than those permitted by $\mathcal{T}$. Thus, if $\langle x, a, y, z \rangle \in \mathcal{T}$ then

[12] *E.g.,* a given amount of coal can generate a certain maximum amount of electricity and a certain minimum amount of smoke.

$\langle x, a, y, z' \rangle \in T$ for all $z' \geq z$.[13] Restrictions of T to various subspaces are denoted by $P(\cdot)$.

Let z_{-k} denote a $(m'-1)$-dimensional quantity vector of all types of emissions other than the k^{th} type of emission. Then we can rewrite the vector z as $\langle z_k, z_{-k} \rangle$. We also define the set

$$\Omega \equiv \left\{ \langle x, a \rangle \in \mathbf{R}_+^{n+1} \;\middle|\; \exists \langle y, z \rangle \in \mathbf{R}_+^{m+m'} \text{ such that } \langle x, a, y, z \rangle \in \mathcal{T} \right\}.$$

Ω is the set of all vectors $\langle x, a \rangle$ of input quantities and cleaning-up levels for which there is some intended production and emission generation, *i.e.,* for which $\mathcal{P}(x, a) \neq \emptyset$. Given the above definitions and notation, Remark 1 follows in an obvious way:

Remark 1: Construction of T from $\mathcal{T}$ implies that

- $\mathcal{P}(x, a) \subseteq P(x, a)$ for all $\langle x, a \rangle \in \Omega$,
- $\Omega = \left\{ \langle x, a \rangle \in \mathbf{R}_+^{n+1} \;\middle|\; P(x, a) \neq \emptyset \right\}$, and
- for all $\langle x, a, y \rangle \in \Omega \times \mathbf{R}_+^m$, we have $P(x, a, y) = P(x, a, y) + \mathbf{R}_+^{m'} = \mathcal{P}(x, a, y) + \mathbf{R}_+^{m'}$.[14]

For all $\langle x, a \rangle \in \Omega$, define the sets

$$\mathcal{Y}(x, a) := \{ y \in \mathbf{R}_+^m \mid \langle y, z \rangle \in \mathcal{P}(x, a) \text{ for some } z \in \mathbf{R}_+^{m'} \},$$

$$\mathcal{Z}(x, a) := \{ z \in \mathbf{R}_+^{m'} \mid \langle y, z \rangle \in \mathcal{P}(x, a) \text{ for some } y \in \mathbf{R}_+^{m} \}, \text{ and}$$

$$Z(x, a) := \{ z \in \mathbf{R}_+^{m'} \mid \langle y, z \rangle \in P(x, a) \text{ for some } y \in \mathbf{R}_+^{m} \}.$$

[13] Vector notation: for any two vectors $a = \langle a_1, \ldots, a_n \rangle$ and $b = \langle b_1, \ldots, b_n \rangle$ in an arbitrary Euclidean space $\mathbf{R}^n$,

$$a \geq b \iff a_i \geq b_i \;\forall\; i = 1, \ldots, n,$$
$$a > b \iff a_i \geq b_i \;\forall\; i = 1, \ldots, n \text{ with } a \neq b, \text{ and}$$
$$a \gg b \iff a_i > b_i \;\forall\; i = 1, \ldots, n.$$

Throughout the paper, quantities of all intended and unintended outputs, inputs, and cleaning-up output are assumed to be non-negative. Where obvious, a vector of zeros will be denoted by 0. However, where the explicit specification of dimensionality becomes important, a vector of zeros of dimension c will be denoted by $0^{(c)}$.

[14] Note the convention: $\emptyset + \mathbf{R}_+^{m'} = \emptyset$.

Remark 2: Construction of T from $\mathcal{T}$ implies that, for all $\langle x, a\rangle \in \Omega$,

$$\mathcal{Y}(x,a) = \{y \in \mathbf{R}^m_+ \mid \langle y, z\rangle \in P(x,a) \text{ for some } z \in \mathbf{R}^{m'}_+\} \text{ and}$$

$$Z(x,a) = \mathcal{Z}(x,a) + \mathbf{R}^{m'}_+.$$

Figures 1 and 2 explain some of the constructs defined above. Both figures assume there is only one intended output and one type of emission. Holding the levels of all inputs and the cleaning-up activity fixed at $\langle x, a\rangle$, these two figures provide two different examples of the restrictions $\mathcal{P}(x,a)$ of $\mathcal{T}$ and $P(x,a)$ of T. Given the vector $\langle x, a\rangle$, y' is the maximum intended output and z' is the minimum level of emission that can be produced. There is also an upper bound z'' on the amount of emission that can be generated given $\langle x, a\rangle$. $\mathcal{P}(x,a)$ is the bounded area $z'ABz''$. On the other hand, the set $P(x,a)$, which is derived as the costly-disposal hull of the set $\mathcal{P}(x,a)$ has an unbounded area. The boundary of $P(x,a)$ emphasises only the lower bound on emission generation. The set $\mathcal{Y}(x,a)$ is the interval $[0, y']$. It is the set of all possible levels of the intended outputs that can be produced under technology $\mathcal{T}$ (and its costly-disposal hull T) by holding inputs and cleaning-up fixed at $\langle x, a\rangle$ and by varying the level of the emission. The set $\mathcal{Z}(x,a)$ is the bounded interval $[z', z'']$. It is the set of all possible levels of the emission that can be produced under technology $\mathcal{T}$, given $\langle x, a\rangle$, by varying the level of the intended output. $Z(x,a)$, the set of emission levels permitted by $P(x,a)$, is the unbounded interval $[z', \infty)$.

3. Axiomatization of EGTs.

In this section we provide some axioms which we believe characterize various types of EGTs. The class of technologies that satisfy these axioms are defined as *by-production* technologies. A preliminary implication, Theorem (BP), of these axioms is also derived.

Axioms (C) and (BOUNDy) place some standard restrictions on $\mathcal{T}$. (C) imposes non-emptiness, convexity, and continuity conditions on $\mathcal{T}$, while (BOUNDy) implies that the quantities of intended outputs that can be produced are bounded when inputs are held fixed and when one allows for some diversion of these fixed inputs for production of a fixed level of cleaning-up activity.[15] Technologies in Figures 1 and 2 satisfy these axioms.

Assumption (C): $\mathcal{T}$ is a non-empty, closed, and convex set.

Assumption (BOUNDy): For all $\langle x, a \rangle \in \Omega$, the set $\mathcal{Y}(x, a)$ is bounded.

Axiom (FDo) imposes standard free-disposability conditions on those intended outputs and inputs that do not cause emissions. Our axiomatization of an EGT also extends output free disposability to emission-causing intended outputs. This is Axiom (FDy_z). Technologies in Figures 1 and 2 satisfy output free disposability of the intended output.

Assumption (FDo): Free output disposability of non-emission causing intended outputs and free input disposability of non-emission causing inputs:

$$\langle x_z, x_o, a, y_z, y_o, z \rangle \in \mathcal{T},\ \bar{x}_o \geq x_o,\ \text{and}\ \bar{y}_o \leq y_o \implies \langle x_z, \bar{x}_o, a, y_z, \bar{y}_o, z \rangle \in \mathcal{T}.$$

Assumption (FDy_z): Free output disposability of emission-causing intended outputs: $\langle x_z, x_o, a, y_z, y_o, z \rangle \in \mathcal{T}$ and $\bar{y}_z \leq y_z \implies \langle x_z, x_o, a, \bar{y}_z, y_o, z \rangle \in \mathcal{T}$.
$(\iff \mathcal{P}(x_z, x_o, a, y_z, y_o) \subseteq \mathcal{P}(x_z, x_o, a, \bar{y}_z, y_o)\ \forall\ \langle x_z, x_o, a, y_z, y_o \rangle \in \Omega \times \mathbf{R}^m_+$ and $\bar{y}_z \leq y_z\ )$.

We consider now the case of emission-causing inputs and the cleaning-up activity. Changes in the levels of these goods affect both intended production and emission generation simultaneously. In particular, an increase in the levels of the emission-causing inputs or a decrease in the level of the cleaning-up activity, increases both the lower bound on emission generation and the upper bound on intended production. The nature of these

[15] Technology $\mathcal{T}$ could permit emissions to be bounded. We could have made an additional assumption: for all $\langle x, a \rangle \in \Omega$, the set $\mathcal{Z}(x, a)$ is bounded. In this paper, we will however be interested only in the lower bounds of $\mathcal{Z}(x, a)$. So upper-bounds on emissions are often ignored in this paper.

changes, when technical inefficiencies in nature's emission generating mechanism and intended production are also taken into account, is captured by Axioms (CCDx_z, a) and (CFDx_z, a).[16]

From the point of view of nature's emission-generating mechanism, standard input and output free-disposability conditions do not apply in the case of emission-generating inputs and the cleaning-up activity – in fact, as in Murty [2010a] and MRL polar opposite conditions, which we call costly disposability of emission-causing inputs and cleaning-up activity, hold: if $\langle \bar{x}, \bar{a} \rangle$ produces $\langle \bar{y}, \bar{z} \rangle$ then, allowing for technical-inefficiencies in nature's emission generating mechanism (*e.g.,* inefficiencies in burning coal), $\bar{z}$ level of emissions could also be produced by *arbitrarily lower* levels of emission-causing inputs, say, x_z, or at *any higher* level of the cleaning-up activity, say, a. Hence, it would *seem* that the set of emissions $P(\bar{x}_z, \bar{x}_o, \bar{a}, \bar{y})$ is a subset of the set of emissions $P(x_z, \bar{x}_o, a, \bar{y})$. *However*, intended production may not allow such a change to be technically feasible: reducing $\bar{x}_z$ or increasing $\bar{a}$ to $\langle x_z, a \rangle$ may mean that too little resources may be available for intended production, so that $\bar{y}$ is no longer feasible for *any* level of emissions. Hence, from the point of view of the overall technology, it is possible that $P(x_z, \bar{x}_o, a, \bar{y}) = \emptyset$. Thus, Axiom (CCD$x_z, a$), below, is only a *conditional* costly-disposability assumption on emission-causing inputs and cleaning-up activity. In Figure 3, it is assumed that there is a single input, and it causes an emission and produces a non-emission generating intended output and a cleaning-up activity. Suppose $\bar{x}$ level of the input is used. Area A in Panel 2 of this figure comprises of various combinations of intended outputs and cleaning-up activity levels that are feasible in intended production given $\bar{x}$. Suppose $\langle \bar{a}, \bar{y} \rangle$ is being produced with $\bar{x}$. Panel 1 shows the nature's emission-generation technology. In particular,

[16] It is convenient to impose these conditions on the costly-disposal hull T rather than directly on $\mathcal{T}$ as these conditions imply some intuitive and empirically observed monotonicity properties of the lower bounds of emission-generation, *e.g.,* lower bounds on emissions increase (resp., decrease) as the levels of emission-causing goods (resp., cleaning-up activity) increase.

holding the cleaning-up activity fixed, the lower bound on emission generation increases with increase in the input. Panel 1 shows that $\bar{x}$ level of input can generate $\bar{z}$ level of the emission. Due to technical inefficiencies in emission generation in nature, $\bar{z}$ level of emission can also be generated by a level of input, say x, *lower* than $\bar{x}$. However, in intended production, the set of feasible intended output and cleaning-up activity levels with x level of the input shrinks to area B (in Panel 2) and $\langle \bar{a}, \bar{y} \rangle$ is no longer producible with the reduced level of the input.

Assumption (CCDx_z, a): Conditional costly disposability of emission-causing inputs and cleaning-up activity:

$$\langle x_z, x_o, a, y_z, y_o, z \rangle \in T,\ \bar{x}_z \leq x_z,\ \bar{a} \geq a,\ \text{and}\ \langle \bar{x}_z, \bar{a} \rangle \neq \langle x_z, a \rangle$$

$$\Longrightarrow\ \text{either}\ P(\bar{x}_z, x_o, \bar{a}, y_z, y_o) = \emptyset\ \text{or}\ P(x_z, x_o, a, y_z, y_o) \subset P(\bar{x}_z, x_o, \bar{a}, y_z, y_o).$$

From the point of view of intended production, standard input and output free disposability conditions apply in the case of emission-causing inputs and the cleaning-up activity: if $\langle x, a \rangle$ produces $\langle y, z \rangle$ then, allowing for technical-inefficiencies in intended production, y level of intended outputs could also be produced by *arbitrarily higher* levels of the first m_z inputs, say, $\bar{x}_z$, or at *any lower* level of the cleaning-up activity, say, $\bar{a}$. Hence, it would seem that the set of intended outputs $P(x_z, x_o, a, z)$ is a subset of the set of intended outputs $P(\bar{x}_z, x_o, \bar{a}, z)$. *However*, nature's emission-generating mechanism may not allow such a change to be technically feasible: increasing x_z or decreasing a to $\langle \bar{x}_z, \bar{a} \rangle$ may mean that emission levels can no longer remain the same, *i.e.*, the levels of emissions that are produced by the new levels of inputs and abatement may be higher than z for *any* level of the intended output. Hence, from the point of view of the overall technology, it is possible that $P(\bar{x}_z, x_o, \bar{a}, z) = \emptyset$. Thus, Axiom (CFD$x_z, a$), below, is only a *conditional* free-disposability assumption on emission-causing inputs and cleaning-up activity. In Figure 4, an increase in the input level from x to $\bar{x}$ expands the the production possibility set of intended production from A to B, so that the combination $\langle a, y \rangle$ is still feasible in intended production with the new input level $\bar{x}$. Nevertheless, in nature, emission level, z, which corresponds to input level x is below the minimum level of emission, $\bar{z}$, generated by input level $\bar{x}$.

Assumption (CFDx_z, a): Conditional free input disposability of emission-causing inputs and conditional free output disposability of cleaning-up activity:

$$\langle x_z, x_o, a, y_z, y_o, z \rangle \in T,\ \bar{x}_z \geq x_z,\ \bar{a} \leq a,\ \text{and}\ \langle \bar{x}_z, \bar{a} \rangle \neq \langle x_z, a \rangle$$
$$\implies \text{either}\ P(\bar{x}_z, x_o, \bar{a}, z) = \emptyset\ \text{or}\ P(x_z, x_o, a, z) \subset P(\bar{x}_z, x_o, \bar{a}, z).$$

A basic result of by-production, Theorem (BP), follows from some of the axioms above. It shows that these axioms imply the empirically observed positive correlation between intended production and emission generation. This correlation is effected through inputs that cause emissions and the cleaning-up activity. As the levels of emission-causing inputs increase or the level of the cleaning-up activity decreases, both the set of feasible levels of intended outputs and the set of feasible levels of emissions shift: in particular ,the upper bounds on intended outputs and the lower bounds on emissions increase. In Figures 5 and 6 it is assumed that all inputs cause the emission. If the input levels increase from x to $\bar{x}$ or cleaning-up level decreases from a to $\bar{a}$, the maximum (resp. minimum) bound on the intended output (resp. emission) increases from y' to $\bar{y}$ (resp. from z' to $\bar{z}$). Thus, the set $\mathcal{Y}(x, a) = [0, y']$ expands to the set $\mathcal{Y}(\bar{x}, \bar{a}) = [0, \bar{y}]$ and the set $Z(x, a) = [z', \infty)$ shrinks to the set $Z(\bar{x}, \bar{a}) = [\bar{z}, \infty)$.

Theorem (BP): *Suppose $\langle x, a \rangle \in \Omega$, $\langle \bar{x}, \bar{a} \rangle \in \Omega$, $\bar{x} \geq x$, $\bar{a} \leq a$, and $\langle x, a \rangle \neq \langle \bar{x}, \bar{a} \rangle$.*

(i) If Assumption (CFDx_z, a) holds then $\mathcal{Y}(x, a) \subset \mathcal{Y}(\bar{x}, \bar{a})$.

(ii) If Assumptions (FDo), (FDy_z), and (CCDx_z, a) hold then $Z(\bar{x}, \bar{a}) \subset Z(x, a)$.

Axiom (INDo) distinguishes between those intended outputs and inputs that cause emissions and those that do not. Ceteris-paribus, technologically feasible changes in the levels of intended outputs or inputs that are non-emission generating have no effect on the lower bounds of emission generation. On the other hand, ceteris-paribus, when there are increases in the levels of intended outputs or inputs that are emission-generating, we expect that the lower-bounds on emission generation will increase. Figure 1 provides an example of a case where the intended output is non-emission generating, while Figure 2 provides a case where it is emission-generating. In Figure 2, the lower bound on emission generation increases from z' (when intended output quantity is 0) to $\overset{*}{z}$ (when intended output quantity increases to $y' > 0$) for fixed levels of all inputs and cleaning-up activity.

Assumption (INDo): The last m_o intended outputs and the last n_o inputs do not cause emissions:

$$\langle x, a\rangle \in \Omega,\ \bar{x}_o \neq x_o,\ \text{and } \bar{y}_o \neq y_o \implies P(x,a,y) = P(x_z, \bar{x}_o, a, y_z, \bar{y}_o)$$
$$\text{if } P(x,a,y) \neq \emptyset \text{ and } P(x_z, \bar{x}_o, a, y_z, \bar{y}_o) \neq \emptyset.$$

It is possible that the emissions generated by a firm when it undertakes intended production (and triggers off nature's emission generating mechanism) can, in turn, affect (generate external effects on) its production of intended outputs. Axioms (INDz), (DETz), and (BENz) distinguish between three such cases.

(INDz) captures the case where such external effects are absent: the set of intended outputs that are feasible under technology $\mathcal{T}$ for given levels of inputs, cleaning-up activity, and emissions is unaffected by changes in its emissions.[17] The technology in Figure 1 satisfies (INDz).

[17] Of course, for this set to be non-empty, it must be the case that we are considering configurations of inputs, cleaning-up, and emissions that are permitted by nature's emission-generating mechanism.

Assumption (INDz): Production of intended outputs is independent of emissions (emissions impose no externalities on intended production):

$$\langle x, a\rangle \in \Omega \text{ and } \bar{z} \neq z \implies \mathcal{P}(x,a,z) = \mathcal{P}(x,a,\bar{z}) \text{ if } \mathcal{P}(x,a,z) \neq \emptyset \text{ and } \mathcal{P}(x,a,\bar{z}) \neq \emptyset.$$

(DETz) (resp. (BENz)) captures the case where emissions generated by a firm impose detrimental (resp. beneficial) external effects of on its intended production: the set of intended outputs that are feasible under technology $\mathcal{T}$ for given levels of inputs, cleaning-up activity, and emissions shrinks (resp. expands) as its emissions increase. Technology in Figure 7 satisfies (DETz), while that in Figure 8 satisfies (BENz). An example of a detrimental external effect is one where smoke from a steel mill can affect its own intended output production by affecting the productivity of its inputs (*e.g.,* labour) in producing steel. Consider the case where a farmer cultivates leguminous plants such as beans and peas along with other crops such as cereals as his intended outputs. Leguminous plants are well-known for attracting nitrogen-fixing bacteria which enrich the soil with nitrogen, which is an important fertilizer in agriculture. This increases the yield of all crops. Thus, this is an example of a case where, in nature, an intended output produces an emission (nitrogen), which imposes a beneficial external effect on intended production.

Assumption (DETz): Emissions impose detrimental externalities on intended production:

$$\langle x, a\rangle \in \Omega,\ \bar{z} \geq z, \text{ and } \bar{z} \neq z \implies \mathcal{P}(x,a,\bar{z}) \subset \mathcal{P}(x,a,z) \text{ if } \mathcal{P}(x,a,z) \neq \emptyset \text{ and } \mathcal{P}(x,a,\bar{z}) \neq \emptyset.$$

Assumption (BENz): Emissions impose beneficial externalities on intended production:

$$\langle x, a\rangle \in \Omega,\ \bar{z} \geq z, \text{ and } \bar{z} \neq z \implies \mathcal{P}(x,a,z) \subset \mathcal{P}(x,a,\bar{z}) \text{ if } \mathcal{P}(x,a,z) \neq \emptyset \text{ and } \mathcal{P}(x,a,\bar{z}) \neq \emptyset.$$

In many instances, there may also be a jointness (simultaneity/complementarity) in production of multiple emissions by a firm. For example, a given variety of impure coal (an input for a thermal electricity producing plant) could have various impurities such as sulphur and nitrogen, besides its carbon content. So when a ton of such coal is burnt, sulphur dioxide, nitrogen oxide, and carbon dioxide are liberated *jointly*. There is no relation of substitutability between the three emissions for a given weight of coal burnt.[18] Rather, there could be a strong complementarity.[19] Allowing for technical inefficiencies in nature's emission-generating mechanism, axiom (Jz) captures this idea. For all $k = 1, \ldots, m'$, ceteris-paribus, the set of feasible levels of the k^{th} type of emission, if non-empty, is not affected by changes in the levels of all other emissions. Figure 9 illustrates this for the case when $m' = 2$. It is assumed that inputs, cleaning-up activity, and intended output levels are fixed at $\langle x, a, y \rangle$. The minimum levels of the two types of emissions produced by nature is given by the vector, $\langle z_1', z_2' \rangle$. There are also upper bounds on emission generation and these are summarized by the vector, $\langle \overset{*}{z}_1, \overset{*}{z}_2 \rangle$. The sets $\mathcal{P}(x, a, y)$ and its costly-disposal hull $P(x, a, y)$ are indicated.

Assumption (Jz): Jointness in the production of pollutants:

$$\forall \ \langle x, a, y \rangle \in \Omega \times \mathbf{R}^m_+,$$

$$P(x_z, x_o, a, y, z_{-k}) \neq \emptyset \text{ and } P(x_z, x_o, a, y, \bar{z}_{-k}) \neq \emptyset \implies$$

$$P(x_z, x_o, a, y, z_{-k}) = P(x_z, x_o, a, y, \bar{z}_{-k}).$$

For convenience, we define the set

$$\mathcal{Z}_{-k}(x, a, y) \equiv \{ z_{-k} \in \mathbf{R}^{m'-1}_+ \mid P(x, a, y, z_{-k}) \neq \emptyset \}.$$

For example, in Figure 9, if $k = 2$ then $Z_{-2}(x, a, y) = [z_1', \infty)$.

[18] *E.g.,* it is not the case that if more sulphur dioxide is produced then less carbon dioxide will be produced, when the amount of coal is held fixed.

[19] *E.g.,* these emissions may be produced in fixed proportions.

Employing the axioms discussed above, we now define a by-production technology, *i.e.,* a technology that simultaneously produces both intended and unintended outputs. Essentially, it is a technology that satisfies the conclusions of Theorem (BP) and, hence, reflects the positive *correlation* between intended production and emission generation.

Definition: $\mathcal{T}$ is a *by-production technology (BPT)* if Assumptions (C), (BOUNDy), (FDo), (FDy_z), (CCDx_z, a), (CFDx_z, a), and (INDo) hold.

Technologies in Figures 1, 2, 7, and 8 are all BPTs. We also distinguish between different types of BPTs depending on whether emissions do or do not impose external effects on intended production:

Definition: $\mathcal{T}$ is a *strong by-production technology (SBPT)* if it is a BPT with Assumption (INDz) holding.

Definition: $\mathcal{T}$ is a *by-production with detrimental externalities technology (DETBPT)* if it is a BPT with Assumption (DETz) holding.

Definition: $\mathcal{T}$ is a *by-production with beneficial externalities technology (BENBPT)* if it is a BPT with Assumption (BENz) holding.

Thus, assuming there is only a single emission type and a single intended output, Figure 1 is an example of a SBPT, Figure 7 is a case of a DETBPT, while Figure 8 is a case of a BENBPT. Suppose, a plot of empirically observed data from an EGT leads to Figure 2. Two possible explanations are possible: (i) Figure 2 reflects a BENBPT and (ii) Figure 2 reflects a technology where, in nature, the intended output is also a cause of emission generation.[20] Hence, based only on the empirically observed data, we may

[20] Strictly speaking, our particular definition of a BENBPT will not classify Figure 2 as being derived from a BENBPT. Nevertheless, it can be argued that our condition for defining a beneficial externality on intended production can be weakened to accommodate Figure 2 as a case of a BENBPT.

not always be able to distinguish between technologies where emissions impose beneficial external effects on intended outputs and technologies where emissions are also caused by intended outputs. Prior engineering and scientific knowledge is required to make such distinctions.

4. Distance-function representations of EGTs.

Based on the various axioms discussed above, which we strongly believe characterize EGTs, we now present a functional representation of such technologies based on the concept of a distance function.

Since an EGT reflects both intended production by firms and the nature's emission-generating mechanism, two distance functions D_1 and D_2 are defined.

D_1 primarily captures the (upper) bounds set by intended production on intended outputs and hence will be defined relative to $\mathcal{T}$ and its restrictions $\mathcal{P}(\cdot)$. Define the mapping

$$D_1 : \Omega \times \mathbf{R}_+^{m+m'} \longmapsto \mathbf{R}_+ \cup \ \{\infty\} \tag{4.1}$$

with image

$$D_1(x, a, y_z, y_o, z) \equiv \inf \ \{\lambda_1 > 0 \ \big| \ \langle \frac{y}{\lambda_1}, \frac{z}{\lambda_1} \rangle \in \mathcal{P}(x, a)\}.$$

D_2 primarily captures the (lower) bounds set by nature on emission generation and hence will be defined relative to the costly-disposal hull T and its restrictions $P(\cdot)$. Define the mapping

$$D_2 : \Omega \times \mathbf{R}_+^{m+m'} \longmapsto \mathbf{R}_+ \cup \ \{\infty\} \tag{4.2}$$

with image

$$D_2(x, a, y_z, y_o, z) := \inf \ \{\lambda_2 \geq 0 \ \big| \ \langle \lambda_2 y, \lambda_2 z \rangle \in P(x, a)\}.$$

For all $\langle x, a, y, z \rangle \in \Omega \times \mathbf{R}_{+}^{m+m'}$, define the constraint sets of the optimization problems (4.1) and (4.2):

$$\begin{aligned} \Lambda_1(x, a, y, z) :=& \{\lambda_1 > 0 \mid \langle \frac{y}{\lambda_1}, \frac{z}{\lambda_1} \rangle \in \mathcal{P}(x, a)\} \text{ and} \\ \Lambda_2(x, a, y, z) :=& \{\lambda_2 > 0 \mid \langle \lambda_2 y, \lambda_2 z \rangle \in P(x, a)\}. \end{aligned}$$

4.1. Properties of mapping D_1.

For any production vector $\langle x, a, y, z \rangle$ (technologically feasible or infeasible), D_1 asks the question, what is the minimum amount by which we need to radially scale down the output vector $\langle y, z \rangle$ to make the resulting vector producible under technology $\mathcal{T}$ with input level x and cleaning-up level a.[21] Thus, $\langle \frac{y}{D_1(x,a,y,z)}, \frac{z}{D_1(x,a,y,z)} \rangle$ is in the set $\mathcal{P}(x, a)$. In Figure 11, this scaling process takes points such as $\langle z, y \rangle$, $\langle z', y' \rangle$, and $\langle \bar{z}, \bar{y} \rangle$ to $\langle \overset{*}{z}, \overset{*}{y} \rangle \in \mathcal{P}(x, a)$. Thus, the minimal scaling-down factors are $D_1(x, a, y, z) = \frac{\|\langle z,y \rangle\|}{\|\langle \overset{*}{z},\overset{*}{y} \rangle\|} < 1$, $D_1(x, a, y', z') = \frac{\|\langle z',y' \rangle\|}{\|\langle \overset{*}{z},\overset{*}{y} \rangle\|} < 1$, and $D_1(x, a, \bar{y}, \bar{z}) = \frac{\|\langle \bar{z},\bar{y} \rangle\|}{\|\langle \overset{*}{z},\overset{*}{a} \rangle\|} > 1$, resp.[22] In each of these cases, the constraint set of optimization (4.1) is non-empty (*e.g.,* $\Lambda_1(x, a, y, z)$ is the set of scaling factors which move $\langle z, y \rangle$ along the ray r), and D_1 is unique and well defined (real valued). However, the constraint set of problem (4.1) can be empty; *e.g.,* take point $\langle \tilde{z}, \tilde{y} \rangle$. No amount of scaling of this vector can make it feasible under $\mathcal{T}$ with $\langle x, a \rangle$. $\Lambda_1(x, a, \tilde{y}, \tilde{z}) = \emptyset$ and by definition of D_1 in (4.1), $D_1(x, a, \tilde{y}, \tilde{z}) = \infty$. Note that, wherever D_1 is well defined, the operation (4.1), takes us to points which are on the (weak) upper-frontier, the curve ABC, of $\mathcal{P}(x, a)$. The points on the upper frontier are reflective of the *upper* bounds on intended production *or* on emission generation.

Theorem (IP) brings out the properties of the mapping D_1 defined with respect to a BPT.

[21] Equivalently, $D_1(x, a, y, z)$ is the inverse of the maximum amount by which $\langle y, z \rangle$ can be radially scaled up so that the resulting vector lies in the set $\mathcal{P}(x, a)$.

[22] $\| \langle z, y \rangle \|$ is the Euclidean length of the vector $\langle z, y \rangle$.

Theorem (IP): *Suppose Assumptions (C), (BOUNDy), (FDo), (CFDx$_z$, a), (CCDx$_z$, a), and (FDy$_z$) hold.*

(i) For all $\langle x, a, y, z\rangle \in \Omega \times \mathbf{R}_+^{m+m'}$, $D_1(x, a, y, z)$ is well defined (finite) and unique if and only if $\Lambda_1(x, a, y, z) \neq \emptyset$.

(ii) $\langle x, a, y, z\rangle \in \mathcal{T} \Rightarrow D_1(x, a, y, z) \leq 1$.

(iii) In the domain where D_1 is well defined:

- *it is homogeneous of degree one and convex, in y and z;*
- *it is non-decreasing in y;*
- *it is non-increasing in x and non-decreasing in a;*
- *it is constant in z if (INDz) holds;*
- *it is non-decreasing in z if (DETz) holds; and*
- *it is non-increasing in z if (BENz) holds.*

(iv) D_1 is continuous in its arguments.

4.2. Properties of mapping D_2.

For any production vector $\langle x, a, y, z\rangle$ (technologically feasible or infeasible), D_2 asks the question, what is the minimum amount by which we need to radially scale up the output vector $\langle y, z\rangle$ to make it feasible to produce under the costly-disposal hull T of technology $\mathcal{T}$ with input level x and cleaning-up level a.[23] Thus, $\langle D_2(x, a, y, z)\, y, D_2(x, a, y, z)\, z\rangle$ is in the set $P(x, a)$. In Figure 12, this scaling process takes points such as $\langle z, y\rangle$, $\langle z', y'\rangle$, and $\langle \bar{z}, \bar{y}\rangle$ to $\langle \hat{z}, \hat{y}\rangle \in \mathcal{P}(x, a)$. Thus, the minimal scaling-up factors are $D_2(x, a, y, z) = \frac{\|\langle \hat{z}, \hat{y}\rangle\|}{\|\langle z, y\rangle\|} < 1$, $D_2(x, a, y', z') = \frac{\|\langle \hat{z}, \hat{y}\rangle\|}{\|\langle z', y'\rangle\|} > 1$, and $D_2(x, a, \bar{y}, \bar{z}) = \frac{\|\langle \hat{z}, \hat{y}\rangle\|}{\|\langle \bar{z}, \bar{y}\rangle\|} < 1$, resp. In each of these cases, the constraint set of optimization (4.2) is non-empty (*e.g.*, $\Lambda_2(x, a, y, z)$ is the set of scaling factor which move $\langle z, y\rangle$ along the ray r), and D_2 is unique and well defined (real valued).

[23] Equivalently, $D_2(x, a, y, z)$ is the inverse of the maximum amount by which we need to radially scale down $\langle y, z\rangle$ so that the resulting vector lies in the set $P(x, a)$.

However, the constraint set of problem (4.2) can be empty; as *e.g.*, $\Lambda_2(x, a, \tilde{y}, \tilde{z}) = \emptyset$ and by definition of D_2 in (4.2), $D_2(x, a, \tilde{y}, \tilde{z}) = \infty$. Note that, wherever D_2 is well defined, the operation (4.2), takes us to points which are on the (weak) lower-frontier AD of $P(x, a)$. The points on the lower frontier are reflective of the *lower* bounds on emission generation imposed by nature.

Theorem (EG) brings out the properties of the mapping D_2 for BPT. Primarily, D_2 characterises the lower bounds on nature's emission-generating mechanism.

Theorem (EG): *Suppose Assumptions (C), (FDo), (FDy$_z$), (CCDx$_z$, a), and (INDo) hold.*

(i) *For all $\langle x, a, y, z\rangle \in \Omega \times \mathbf{R}^{m+1}_+$, $D_2(x, a, y, z)$ is well defined (finite) and unique if and only if $\Lambda_2(x, a, y, z) \neq \emptyset$.*

(ii) $\langle x, a, y, z\rangle \in \mathcal{T} \Rightarrow D_2(x, a, y, z) \leq 1$.

(iii) *In the domain where D_2 is well defined:*

- *it is homogeneous of degree one and convex in y and z;*
- *it is non-increasing in z and non-decreasing in y_z;*
- *it is non-decreasing in x and non-increasing in a; and*
- *it is constant in y_o and x_o.*

(iv) *D_2 is continuous in its arguments.*

4.3. A representation theorem for a BPT and its efficient frontier.

Theorem (BP-REPR), below, shows that, provided a technical condition that we call Assumption $(*)$ holds, a BPT can be represented functionally by employing the distance functions D_1 and D_2.

Assumption $(*)$: For all $\langle x, a\rangle \in \Omega$,

$$z \in Z(x, a) \text{ and } z \notin \mathcal{Z}(x, a) \implies \kappa z \notin \mathcal{Z}(x, a) \;\forall\; \kappa \geq 1.$$

For a BPT, Assumption ($*$) is a condition on the upper-bounds of emission generation. It is satisfied, *e.g.,* for the technology in Figure 10, where it is assumed that there are two types of emissions and axiom (Jz) holds. Similarly, it can be seen that it is also satisfied for technologies in Figures 1, 2, 7, and 8, where we assume that there is only one type of emission and $\mathcal{Z}(x, a)$ is bounded for all $\langle x, a\rangle \in \Omega$.

Theorem (BP-REPR): *Suppose $\mathcal{T}$ is a BPT and Assumption ($*$) holds. Then*

$$\langle x, a, y, z\rangle \in \mathcal{T} \iff D_1(x, a, y, z) \leq 1 \text{ and } D_2(x, a, y, z) \leq 1.$$

The necessity part of the above representation theorem is clear from the definitions of D_1 and D_2 in (4.1) and (4.2): if $\langle x, a, y, z\rangle \in \mathcal{T}$, then clearly $1 \in \Lambda_1(x, a, y, z)$ and $1 \in \Lambda_2(x, a, y, z)$, so that $D_1(x, a, y, z) \leq 1$ and $D_2(x, a, y, z) \leq 1$.

Sufficiency can be intuitively explained by employing Figures 11 and 12 to illustrate the contrapositive: for a production vector that is *not* feasible under a given BPT $\mathcal{T}$, at least one of the distance functions D_1 or D_2 takes a value (strictly) bigger than 1. Three cases are possible:

(1) The point could be such as $\langle \bar{z}, \bar{y}\rangle \notin \mathcal{P}(x, a)$ for which $D_2(x, a, \bar{y}, \bar{z}) = \frac{\|\langle \hat{z}, \hat{y}\|}{\|\langle \bar{z}, \bar{y}\|} \leq 1$. In that case, $D_1(x, a, \bar{y}, \bar{z}) = \frac{\|\langle \bar{z}, \bar{y}\|}{\|\langle \overset{*}{z}, \overset{*}{y}\|} > 1$.

(2) The point could be such as $\langle z', y'\rangle \notin \mathcal{P}(x, a)$ for which $D_1(x, a, y', z') = \frac{\|\langle z', y'\|}{\|\langle \overset{*}{z}, \overset{*}{y}\|} \leq 1$. In that case, $D_2(x, a, \bar{y}, \bar{z}) = \frac{\|\langle \hat{z}, \hat{y}\|}{\|\langle z', y'\|} > 1$.

(3) The point could be such as $\langle \tilde{z}, \tilde{y}\rangle \notin \mathcal{P}(x, a)$ for which $D_1(x, a, \tilde{y}, \tilde{z}) = D_2(x, a, \tilde{y}, \tilde{z}) = \infty > 1$.

Theorem (BP-EFFICIENCY), below, provides a characterization of the efficient frontier of a BPT. We first define the notion of efficiency.[24]

[24] This definition of efficiency is normatively acceptable for the case of detrimental emissions. It will need to be modified for the case of beneficial emissions, where efficiency would require there exists no other point $\langle \bar{x}, \bar{a}, \bar{y}, \bar{z}\rangle \in \mathcal{T}$ such that $\langle x, a, y, z\rangle \neq \langle \bar{x}, \bar{a}, \bar{y}, \bar{z}\rangle$, $\bar{x} \leq x$, $\bar{a} \geq a$, $\bar{y} \geq y$, and $\bar{z} \geq z$.

Definition: Suppose $\mathcal{T}$ is a BPT. $\langle x, a, y, z\rangle$ is a *strictly efficient point of* $\mathcal{T}$ if $\langle x, a, y, z\rangle \in \mathcal{T}$ and there exists no other point $\langle \bar{x}, \bar{a}, \bar{y}, \bar{z}\rangle \in \mathcal{T}$ such that $\langle x, a, y, z\rangle \neq \langle \bar{x}, \bar{a}, \bar{y}, \bar{z}\rangle$, $\bar{x} \leq x$, $\bar{a} \geq a$, $\bar{y} \geq y$, and $\bar{z} \leq z$. The strictly efficient frontier of $\mathcal{T}$, denoted by $Front(\mathcal{T})$, is the set of strictly efficient points of $\mathcal{T}$.

In each of Figures 1 and 7, there is only one strictly efficient point in $\mathcal{P}(x, a)$, namely, the point A. It can be verified that at such points both D_1 and D_2 take value equal to 1. Theorem (BP-EFFICIENCY) shows that this is true generally for the cases of SBPTs and DETBPTs when no intended output is emission-generating. In each of Figures 2 and 8, (which are the cases where the intended output is emission generating or where the technology is BENBPT), the set of strictly efficient points in $\mathcal{P}(x, a)$ is the line-segment Az'. At all such points, it can be verified that D_2 takes value equal to 1. However, this is not true of D_1.

Theorem (BP-EFFICIENCY): *Suppose $\mathcal{T}$ is a SBPT or a DETBPT, $m_z = 0$, Assumption $(*)$ holds, and D_1 and D_2 are defined as in (4.1) and (4.2), resp. Then:*

$$\langle x, a, y, z\rangle \textit{ is a strictly efficient point of } \mathcal{T} \iff D_1(x, a, y, z) = 1 \textit{ and } D_2(x, a, y, z) = 1.$$

4.4. Decomposition of a BPT into intended-production and nature's emission-generation sets.

Suppose we observe empirically a BPT $\mathcal{T}$. The observed feasible production points must be simultaneously feasible with respect to the underlying intended production technology and nature's emission-generation set. Can we, employing this data, independently recover these two sets? Theorem (BP-DECOMP) shows that by applying programmes (4.1) and (4.2) we can decompose $\mathcal{T}$ into its underlying intended production technology $\mathcal{T}_1$ and the costly-disposal hull of the nature's emission-generation set T_2. If, in addition, Assumption (Jz) is true and no intended output is emission generating in nature, then we can also obtain distinct emission-generating sets of nature for different emission types by defining new distance functions.

Theorem (BP-DECOMP): *Suppose $\mathcal{T}$ is a BPT, Assumption ($*$) holds, and D_1 and D_2 are defined as in (4.1) and (4.2), resp. Define the intended production technology*

$$\mathcal{T}_1 := \{\langle x, a, y, z\rangle \in \Omega \times \mathbf{R}_+^{m+m'} \mid D_1(x, a, y, z) \leq 1\}$$

and the costly disposal-hull of nature's emission-generation set

$$T_2 := \{\langle x, a, y, z\rangle \in \Omega \times \mathbf{R}_+^{m+m'} \mid D_2(x, a, y, z) \leq 1\}.$$

Then $\mathcal{T} = \mathcal{T}_1 \cap T_2$.

Suppose, in addition, $m_z = 0$ and (Jz) is true. For all $k = 1, \ldots, m'$, define the mappings

$$D_{2k} : \Omega \times \mathbf{R}_+^{m+m'} \longmapsto \mathbf{R}_+ \cup \ \{\infty\}$$

with image

$$D_{2k}(x, a, y, z_k, z_{-k}) = \inf\{\lambda_{2k} > 0 \mid \lambda_{2k} z_k \in P(x, a, y, z_{-k})\}. \tag{4.3}$$

Then, in the domain where it is well defined, each D_{2k} is a real-valued function, which is constant in z_{-k} and y. For all $k = 1, \ldots, m'$, define the costly-disposal hulls of nature's separate emission-generation sets

$$T_{2k} := \{\langle x, a, y, z\rangle \in \Omega \times \mathbf{R}_+^{m+m'} \mid D_{2k}(x, a, y, z_k, z_{-k}) \leq 1\}.$$

Then $T_2 = \bigcap_{k=1}^{m'} T_{2k}$.

In Figure 13, we assume, $m' = 2$. The technology satisfies (Jz). Suppose $k = 2$. As seen, $P(x, a, z_1') = P(x, a, z_1^{''})$. D_{22} can be defined as in (4.3). Wherever well defined, it is constant in z_1, *e.g.,* $D_{22}(x, a, y, z_2^{''}, z_1^{''}) = D_{22}(x, a, y, z_2^{''}, z_1^*) = \frac{z_2'}{z_2^{''}}$. Similarly, we can also define D_{21}, *e.g.,* $D_{21}(x, a, y, z_1^{''}, z_2^{''}) = \frac{z_1'}{z_1^{''}}$. D_{21} is constant in z_2, wherever it is well defined.

5. Constructing BPTs.

In the above discussion, we began with a (possibly, empirically observed) BPT and then derived it's functional representation by employing programmes (4.1) and (4.2). Alternatively, we can construct a BPT by beginning with two real-valued continuous *functions* D_1 and D_2 defined on the domain $\mathbf{R}_+^{n+m+m'+1}$ with properties listed in parts (iii) of Theorems (IP) and (EG), resp., and use them to construct a BPT set, say $\mathcal{T}$.[25]

Theorem (BP-CONSTRUCT): *Suppose the mappings*

$$D_1 : \mathbf{R}_+^{n+m+m'+1} \longmapsto \mathbf{R}_+$$

with image $\lambda_1 = D_1(x, a, y_z, y_o, z)$ *and*

$$D_2 : \mathbf{R}_+^{n+m+m'+1} \longmapsto \mathbf{R}_+$$

with image $\lambda_2 = D_2(x, a, y_z, y_o, z)$ *are two continuous functions such that*

(1) D_1 *is homogeneous of degree one and convex in* y *and* z*, non-decreasing in* y*, non-increasing in* x*, and non-decreasing in* a *and*

(2) D_2 *is homogeneous of degree one and convex in* y *and* z*, non-increasing in* z_k*, non-decreasing in* y_z*, non-decreasing in* x*, non-increasing in* a*, and constant in* y_o *and* x_o*.*

Define the sets

$$\mathcal{T}_1 := \{\langle x, a, y, z\rangle \in \mathbf{R}_+^{n+m+m'+1} \mid D_1(x, a, y, z) \leq 1\},$$

$$T_2 := \{\langle x, a, y, z\rangle \in \mathbf{R}_+^{n+m+m'+1} \mid D_2(x, a, y, z) \leq 1\}, \text{ and}$$

$$\mathcal{T} = \mathcal{T}_1 \cap T_2.$$

$\mathcal{T}$ *is a*

(i) SBPT if D_1 *is constant in* z*.*

[25] $\mathcal{T}$ so derived may not have upper bounds on emissions. However, if we had also focussed on the upper bounds on nature's emission generation and had discussed a third class of distance functions which capture these upper-bounds, then $\mathcal{T}$ would also have been bounded in z.

(ii) DETBPT if D_1 is non-decreasing in z.

(i) BENBPT if D_1 is non-increasing in z.

Example 1.

In the following example, we construct a BPT $\mathcal{T}$ by applying Theorem (BP-CONSTRUCT). Consider the case where a firm employs wood, water, chemicals, labor, and capital as inputs to produce paper. Two types of residuals are jointly produced as by-products. Suppose the firm also has an effluent treatment plant. Thus, the quantity vectors are $z = \langle z_1, z_2 \rangle$, $x_z = \langle x_{z_1}, x_{z_2} \rangle$, and $x_o = \langle x_{o_1}, x_{o_2}, x_{o_3} \rangle$, where the first and second emission-causing inputs are wood and chemicals, resp. Consider the following distance functions:

$$
\begin{aligned}
D_1(x_{z_1}, x_{z_2}, x_{o_1}, x_{o_2}, x_{o_3}, a, y, z) &= \frac{\left(\prod_{i=1}^{2} x_{z_i}^{\frac{1}{5}}\right)\left(\prod_{i=1}^{3} x_{o_i}^{\frac{1}{5}}\right)}{y + a}, \\
D_{21}(x_{z_1}, x_{z_2}, x_{z_3}, x_{z_4}, x_{z_5}, a, y, z) &= \frac{\alpha_{11} x_{z_1} + \alpha_{12} x_{z_2} - \theta a}{z_1}, \\
D_{22}(x_{z_1}, x_{z_2}, x_{z_3}, x_{z_4}, x_{z_5}, a, y, z) &= \frac{\alpha_{21} x_{z_1} + \alpha_{22} x_{z_2} - \theta a}{z_2}, \text{ and} \\
\alpha_{11} > 0,\ \alpha_{12} > 0,\ \alpha_{21} > 0,&\ \alpha_{22} > 0,\ \theta > 0.
\end{aligned}
$$

Define:

$$
\begin{aligned}
\mathcal{T}_1 &:= \{\langle x, a, y, z \rangle \in \mathbf{R}_+^9 \mid y \le \left(\prod_{i=1}^{2} x_{z_i}^{\frac{1}{5}}\right)\left(\prod_{i=1}^{3} x_{o_i}^{\frac{1}{5}}\right) - a\}, \\
T_{21} &:= \{\langle x, a, y, z \rangle \in \mathbf{R}_+^9 \mid z_1 \ge \alpha_{11} x_{z_1} + \alpha_{12} x_{z_2} - \theta a\}, \\
T_{22} &:= \{\langle x, a, y, z \rangle \in \mathbf{R}_+^9 \mid z_2 \ge \alpha_{21} x_{z_1} + \alpha_{22} x_{z_2} - \theta a\}, \text{ and} \\
\mathcal{T} &= \mathcal{T}_1 \cap T_2, \text{ where } T_2 = T_{21} \cap T_{22}.
\end{aligned}
$$

It can be verified that $\mathcal{T}$, which could be a potential technology of the firm, is a SBPT that satisfies (Jz). Further, $m_z = 0$. $\mathcal{T}_1$ and T_2 are the underlying intended production technology and the (costly-disposal hull of) nature's emission generation technology, resp.

From Theorem (BP-EFFICIENCY) it follows that

$$\begin{aligned} Front(\mathcal{T}) \equiv \Big\{ \ \langle x, a, y, z \rangle \in \mathbf{R}^8_+ \ \Big| \ & (i) \ y = \left(\prod_{i=1}^{2} x_{z_i}^{\frac{1}{5}} \right) \left(\prod_{i=1}^{3} x_{o_i}^{\frac{1}{5}} \right) - a, \\ & (ii) \ z_1 = \alpha_{11} x_{z_1} + \alpha_{12} x_{z_2} - \theta a, \ \text{and} \\ & (iii) \ z_2 = \alpha_{21} x_{z_1} + \alpha_{22} x_{z_2} - \theta a \ \Big\}. \end{aligned} \tag{5.1}$$

Note, there are several abatement strategies available to a firm, such as decreasing the use of the emission causing inputs, wood or paper, or increasing the level of the cleaning-up activity, or some combinations of these. Different abatement strategies generate different trade-offs (correlations) between intended production and emission generation along $Front(\mathcal{T})$, which can be computed using the implicit function theorem, *e.g.,*

- Consider changing the level of the emission-causing chemical input to $\bar{x}_{z_2}$ holding the levels all other inputs and the cleaning-up activity fixed. Employing the implicit function theorem[26], yields trade-off between y and z_1 as

$$\frac{\partial y}{\partial z_1} = \frac{\left(\prod_{i=1}^{3} x_{o_i}^{\frac{1}{5}} \right) x_{z_1}^{\frac{1}{5}} x_{z_2}^{-\frac{4}{5}}}{5\alpha_{12}} > 0.$$

- Ceteris paribus, if the level of the emission-causing wood input is changed, then the trade-off between y and z_1 is

$$\frac{\partial y}{\partial z_1} = \frac{\left(\prod_{i=1}^{3} x_{o_i}^{\frac{1}{5}} \right) x_{z_2}^{\frac{1}{5}} x_{z_1}^{-\frac{4}{5}}}{5\alpha_{11}} > 0.$$

- Ceteris paribus, if the level of cleaning-up activity is changed, then the trade-off between y and z_1 is

$$\frac{\partial y}{\partial z_1} = \frac{1}{\theta} > 0.$$

[26] Which amounts to substituting for x_{z_2} from (ii) of (5.1) into (i) of (5.1).

6. Application of by-production modeling of emission-generating technologies to international non-cooperative emission outcomes.

In this section, the distance function representation of a BPT obtained in the previous sections is employed to help study and explain cross-country differences in emission levels at a non-cooperative Nash equilibrium of a global economy. Richness or poorness of a country is attributed solely to differences in the initial endowments of a particular resource, which we call capital. The aim is to study to what extent the sheer differences in the availability of a key productive factor can help explain cross-country differences in emissions. All countries, rich or poor, are assumed to have the same preferences for a clean environment (or dirty emissions) and consumption of an intended output. In particular, emission is assumed to be an inferior good. Each country is free to choose what to produce (subject only to its capital constraint) from a set of technological choices, which is also assumed to be common to all countries.

6.1. Specification of the global economy.

(G1) to (G7) are the specifications of the global economy we will study. We would like to model all standard aspects of production theory: in particular, the economic law of diminishing return. The BP approach allows us to do so in a most general way. However, a very general specification of a BPT can make the analysis quite intractable. Hence, the structure of the technology described below is kept reasonably simple. Yet, it encompasses a rich set of production features which are realistic, quite intuitive, and pertinent to the question at hand.

(G1) S countries indexed by s.

(G2) Every country employs inputs capital, labour, and n_z varieties of fuels[27] with varying emission intensities[28] to produce a non-emission causing intended output and a

[27] *e.g.,* coal, oil, *etc.*.

[28] Emissions per unit of fuel burnt.

cleaning-up activity. Hence, $m = 1$, $m' = 1$, $n = n_o + n_z$, $n_o = 2$, $m_z = 0$. For notational convenience, let k denote the quantity of capital input, l the quantity of labor input, and $x_z = \langle x_{z_1}, \ldots, x_{z_{n_z}} \rangle$ the quantity vector of various varieties of fuels used in the production of the intended output. Thus, $x = \langle x_z, x_o \rangle$ with $x_o = \langle l, k \rangle$.

(G3) It is also possible that, in addition to contributing directly to intended production, fuel inputs may also be required to run capital equipment; *i.e.,* there is also a strong complementarity between the use of the fuel inputs and capital, *e.g.,* coal and other fuels may be needed directly to produce thermal electricity. In addition it, along with other fuel inputs such as oil, may also be needed to provide power for running the capital equipment in a thermal power plant. For simplicity, let us assume that the use of capital input requires only the first type of fuel. Define a smooth function $\phi : \mathbf{R}_+ \longrightarrow \mathbf{R}_+$ with image $\phi(k)$ as the amount of the first type of fuel needed to run k units of capital equipment. We will consider two extreme cases: one where the use of capital is not fuel intensive and the other where it is. Thus, function ϕ is such that either $\phi(k) = 0 \ \forall \ k$ or $\phi_k(k) > 0 \ \forall \ k$.

(G4) All countries have identical technologies represented by smooth distance functions $D_1 : \mathbf{R}_+^{4+n_z} \longrightarrow \mathbf{R}_+$ and $D_2 : \mathbf{R}_+^{4+n_z} \longrightarrow \mathbf{R}_+$ with images[29]

$$D_1(x_z, l, k, a, y) = \frac{h(a, y)}{f(x_z, l, k)} \quad \text{and}$$

$$D_2(x_z, l, k, a, y) = \frac{\sum_{i=1}^{n_z} \alpha_i x_{z_i} + \alpha_1 \phi(k) - \theta a}{z}.$$

f is a smooth function, which is concave in x_z and l with

$f_{x_{z_i}} > 0, \ f_l > 0, \ f_k > 0,$

$f_{x_{z_i} x_{z_i}} \leq 0, \ f_{x_{z_i} x_{z_{\hat{\imath}}}} = 0, \ f_{ll} = 0, \ f_{lk} = 0, \ f_{l x_{z_i}} = 0, \ f_{x_{z_i} k} = 0, \ i, \hat{\imath} = 1, \ldots, n_z, \ i \neq \hat{\imath},$

h is a smooth and quasi-convex function with $h_a > 0, \ h_y = 1, \ h_{ya} = 0, \ h_{aa} \geq 0,$ and $\alpha_i > 0, \ \theta > 0.$

[29] Note that capital itself is a non-emission causing input in nature. It is associated with emission only to the extent the fuel input, $\phi(\cdot)$, needed for running it triggers-off the nature's emission-causing mechanism. Hence, the fuel input, $\phi(\cdot)$, is a part of the nature's emission-causing mechanism.

For all $i = 1, \ldots, n_z$, α_i denotes the emission intensity of i^{th} variety of coal. Let θ denote the cleaning-up intensity of the cleaning-up activity.[30] The universally available technology is denoted by $\mathcal{T}$:[31]

$$\begin{aligned}\mathcal{T} &= \big\{ \ \langle x_z, l, k, a, y, z\rangle \in \mathbf{R}_+^{5+n_z} \ \big| \ D_1(x_z, l, k, a, y) \leq 1, \ D_2(x_z, l, k, a, z) \leq 1 \big\} \\ &= \big\{ \ \langle x_z, l, k, a, y, z\rangle \in \mathbf{R}_+^{5+n_z} \ \big| \ h(a, y) \leq f(x_z, l, k), \ z \geq \sum_{i=1}^{n_z} \alpha_i x_{z_i} + \alpha_1 \phi(k) - \theta a \big\}.\end{aligned} \tag{6.1}$$

(G5) For every variety of fuel, there is a constant marginal (labour) cost of extraction (production), which is common for all countries: For all $i = 1, \ldots, n_z$, the labour extraction cost of one unit of the i^{th} variety of fuel for any country is $c_i > 0$. If x_z is the amount of fuel used directly in the production of the intended outputs, the use of capital input is fuel intensive, and k is the amount of capital input used, then the total labour cost of extraction is given by the function $\varphi : \mathbf{R}_+^{2+n_z} \longrightarrow \mathbf{R}_+$ with image $\varphi(x_z, l, k) \equiv l + \sum_{i=1}^{n_z} c_i x_{z_i} + c_1 \phi(k)$.

(G6) All countries have the same initial endowment, $\bar{L}$, of labour input. Countries differ with respect to the endowment of capital. There is autarky. Capital and labour are fixed inputs. Differences in technological relations across countries arise only because of the differences in their capital endowments. Capital endowment of country s is k^s. Thus, technology of country s is

$$\mathcal{T}^s = \big\{ \ \langle x_z, l, k, a, y, z\rangle \in \mathbf{R}_+^{5+n_z} \ \big| \ \langle x_z, l, k, a, y, z\rangle \in \mathcal{T}, \ k = k^s, \ \text{and } l \leq \bar{L} \ \big\}. \tag{6.2}$$

(G7) Welfare of a country depends on its consumption of the intended output, its own emission, and global emissions. All countries have identical preferences. Global emissions are the sum of emissions by all countries. For every country $s = 1, \ldots, S$, assuming $\tilde{g}$

[30] Cleaning-up intensity is the reduction in emissions per unit of cleaning-up activity.

[31] Since the upper bounds on emission generation have not been specified, note that the costly-disposal hull of $\mathcal{T}$ is also $\mathcal{T}$. Hence, restriction mappings $\mathcal{P}(\cdot)$ of $\mathcal{T}$ and $P(\cdot)$ of the costly-disposal hull of $\mathcal{T}$ are equal.

denotes the amount of emissions of all but the s^{th} country, preferences are represented by a smooth and strictly quasi-concave utility function $u : \mathbf{R}^3_+ \longrightarrow \mathbf{R}_+$ with image

$$\begin{aligned} & u = u(y, z, \tilde{g} + z) \\ & u_y > 0,\ u_z < 0,\ u_{\tilde{g}} < 0, u_{yz} = 0,\ u_{y\tilde{g}} = 0,\ u_{\tilde{g}z} = 0,\ u_{\tilde{g}\tilde{g}} = 0. \end{aligned} \tag{6.3}$$

The form of u suggests that emission of a country has both local and global effects. In particular, the latter has only a first-order effect on the welfare of a country, while the former can have both first and second order welfare effects. Emissions cause disutility to a country.

Remark 3: Under specifications (G1) to (G7) of the global economy, D_1 (resp., D_2) satisfies all properties in (1) (resp., (2)) of Theorem (BP-CONSTRUCT). Hence, for all $s = 1, \ldots, S$, $\mathcal{T}^s$ is a BPT.

6.2. The optimization problem of an economy and a non-cooperative Nash equilibrium emission outcome.

First, let us define a constant ρ such that

$$\begin{aligned} \rho &= \phi^{-1}(\frac{\bar{L}}{c_1}) && \text{if } \phi \text{ is smooth and increasing in } k \text{ and} \\ &= \infty && \text{if } \phi(k) = 0 \text{ for all } k. \end{aligned}$$

If the use of capital requires fuel then there is an upper bound on the capital stock that a country can use, since the maximum amount of labour that can be put to fuel extraction purposes is bounded by $\bar{L}$. This upper bound is ρ. So in our analysis, $k \in (0, \rho)$. Define the function $U : \mathbf{R}_{++} \times (0, \rho) \longrightarrow\ \mathbf{R}_+$ with image

$$\begin{aligned} U(\tilde{g}, k) \equiv \max_{x_z > 0,\ l > 0,\ a > 0,\ y > 0,\ z > 0} \ & u(y, z, \tilde{g} + z) \\ & \text{subject to} \\ D_1(x_z, l, k, a, y) \leq 1,\ D_2(x_z, l, k, a, z) \leq 1,\ & \text{and } \varphi(x_z, l, k) \leq \bar{L}. \end{aligned} \tag{6.4}$$

Every country with a particular initial endowment k of capital and facing emissions $\tilde{g}$ from all other countries solves problem (6.4). Let the solution mapping to (6.4) be denoted by

$\Phi : \mathbf{R}_{++} \times (0, \rho) \longrightarrow \mathbf{R}_{++}^{4+n_z}$ with image $\langle x_z, l, a, y, z \rangle \in \Phi(\tilde{g}, k)$. If the mapping Φ is a function, then $\Phi(\tilde{g}, k) \equiv \left\langle \hat{x}_z(\tilde{g}, k),\ \hat{l}(\tilde{g}, k),\ \hat{a}(\tilde{g}, k),\ \hat{y}(\tilde{g}, k),\ \hat{z}(\tilde{g}, k) \right\rangle$. The following theorem shows that Φ is well-defined. In particular, every configuration $\langle \tilde{g}, k \rangle \in \mathbf{R}_{++} \times (0, \rho)$ results in a unique choice of intended-output and emission levels.

Theorem (SOLUTION-OPT): *Under specifications (G1) to (G7) of the global economy, the solution mapping Φ of (6.4) is well-defined and upper-hemicontinuous. In particular, the components y and z of Φ are continuous functions $\hat{y}$ and $\hat{z}$.*

Define the function $\tilde{g}^s : \mathbf{R}_{++}^{S-1} \longrightarrow \mathbf{R}_{++}$ for $s = 1, \ldots, T$, with images[32]

$$\tilde{g}^s(z^{(-s)}) = \sum_{s' \neq s} z^{s'}. \tag{6.5}$$

Definition: $\langle \overset{*}{z}^1, \ldots, \overset{*}{z}^S \rangle$ is a *non-cooperative Nash equilibrium emission outcome of the global economy with specifications (G1) to (G7)* if, for all $s = 1, \ldots, S$, we have

$$\overset{*}{z}^s = \hat{z}\left(\tilde{g}^s(\overset{*}{z}^{(-s)}),\ k^s\right).$$

Remark 4: There exists a unique non-cooperative Nash equilibrium emission outcome of the global economy with specifications (G1) to (G7).[33]

The following theorem examines the relation between the levels of capital endowment and emission at a non-cooperative Nash equilibrium. It shows that this relation crucially depends on the monotonicity property of function $\hat{z}$ in k. As we move from capital-poor to capital-rich countries, emissions decrease or increase depending on whether $\hat{z}$ is non-increasing or non-decreasing in k. On the other hand, if $\hat{z}$ has a global maximum at $\overset{*}{k}$ level of capital, then we obtain an inverted U shape curve of emissions–a manifestation of the environmental Kuznets curve phenomenon.

[32] For any $s = 1, \ldots, S$, the vector $z^{(-s)}$ denotes a $(S-1)$-dimensional quantity vector of emissions generated by all countries other than the s^{th} country.

[33] Proof is standard.

Theorem (NON-COOP-NASH k, z RELATION): *Let $\langle \overset{*}{z}^1, \ldots, \overset{*}{z}^S \rangle$ be a non-cooperative Nash equilibrium emission outcome of the global economy with specifications (G1) to (G7). Suppose $k^1 \leq k^2 \leq \ldots \leq k^S$.*

(i) If the function $\hat{z}$ is non-increasing (resp., non-decreasing) in k and constant in $\tilde{g}$ then $\overset{*}{z}^1 \geq \overset{*}{z}^2 \geq \ldots \geq \overset{*}{z}^S$ (resp., $\overset{*}{z}^1 \leq \overset{*}{z}^2 \leq \ldots \leq \overset{*}{z}^S$).

(ii) If the function $\hat{z}$ is constant in $\tilde{g}$ and there exists $\overset{*}{k} \geq 0$ such that the function $\hat{z}$ has a maximum at $\overset{*}{k}$ then, for any $\tilde{g}$, we have $\overset{*}{z}^1 \leq \overset{*}{z}^2 \leq \ldots \leq \overset{*}{z}^{s-1} \leq \hat{z}(\tilde{g}, \overset{*}{k})$ and $\hat{z}(\tilde{g}, \overset{*}{k}) \geq \overset{*}{z}^s \geq \overset{*}{z}^{s+1} \geq \ldots \geq \overset{*}{z}^S$, where $s \equiv \min\{s' \leq T \mid k^{s'} \geq \overset{*}{k}\}$.

6.3. A comparative static exercise.

Theorem (NON-COOP-NASH k, z RELATION) implies that the knowledge of the monotonicity property of function $\hat{z}$ in k is important for understanding the cross-country differences in emission levels at a non-cooperative Nash equilibrium. This calls for a comparative static analysis of problem (6.4), which is the agenda of the current section. The Lagrangian of (6.4) is

$$\begin{aligned} L = u(y, z, \tilde{g} + z) \; - \lambda[h(y, a) - f(x_z, l, k)] \; &- \delta[\sum_{i=1}^{n_z} \alpha_i x_{z_i} + \alpha_1 \phi(k) - \theta a - z] \\ &- \gamma[l + \sum_{i=1}^{n_z} c_i x_{z_i} + c_1 \phi(k) - \bar{L}]. \end{aligned} \tag{6.6}$$

The first-order conditions of (6.4) for an interior optimum are

$$u_y - \lambda = 0, \tag{6.7}$$

$$u_z + \delta = 0, \tag{6.8}$$

$$-\lambda h_a + \delta\theta = 0, \tag{6.9}$$

$$\lambda f_l - \gamma = 0, \tag{6.10}$$

$$\lambda f_{x_{z_i}} - \delta\alpha_i - \gamma c_i = 0, \; \forall \; i = 1, \ldots, n_z, \tag{6.11}$$

$$-[h(y,a) - f(x_z,l,k)] = 0, \tag{6.12}$$

$$-[\sum_{i=1}^{n_z} \alpha_i x_{z_i} + \alpha_1 \phi(k) - \theta a - z] = 0, \text{ and} \tag{6.13}$$

$$-[l + \sum_{i=1}^{n_z} c_i x_{z_i} + c_1 \phi(k) - \bar{L}] = 0. \tag{6.14}$$

(6.7), (6.8), and (6.9) yield

$$-\frac{u_z}{u_y} = \frac{h_a}{\theta}. \tag{6.15}$$

Note, at an optimum to (6.4), (6.7) implies that $\lambda > 0$. Hence, (6.9), (6.10), and (6.11) yield[34]

$$\frac{f_{x_{z_i}} - f_l c_i}{\alpha_i} = \frac{h_a}{\theta} \equiv \frac{\frac{h_a}{h_y}}{\theta}, \qquad \forall\ i = 1, \ldots, n_z. \tag{6.16}$$

The left-side of (6.16) is the trade-off between intended production and emission generation, obtained as a result of increasing the usage of the i^{th} fuel: the numerator is the *net* increase in intended production,[35] while the denominator is the increase in the level of emission. The right-side of (6.16) is the trade-off between intended production and emission generation due to a decrease in the level of the cleaning-up activity: the numerator is the increase in intended production, while the denominator is the increase in the level of emission. At a solution to (6.4), for a given level of capital, (6.16) shows that the trade-offs between intended production and emission generation obtained by *any* abatement strategy (*e.g.,* decrease in the level of any fuel input or an increase in the level of cleaning-up activity) are equalized. (6.16), thus, describes the trade-off in production at a solution to (6.4). (6.15) shows that the trade-offs between intended-output and emission in consumption and production are also equalized at a solution to (6.4). The trade-off between intended-output and emission in consumption is the marginal rate of substitution (MRS) between consumption of the intended output and emission.

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[34] Recall, $h_y = 1$.

[35] It is net of the loss due to diverting labour from intended production to extraction of the additional amount of the fuel.

Standard comparative static analysis of problem (6.4) yields the following Lemma. The Lemma yields general results, but we will be more interested in employing this lemma to study some interesting special cases later.

Lemma (COMP-STATICS$k, \tilde{g}$): *Under specifications (G1) to (G7) of the global economy, if the solution mapping Φ of (6.4) is a differentiable function, $h_{aa} > 0$, and $f_{x_{z_i} x_{z_i}} < 0$ for $i = 1, \ldots, n_z$, then Φ_k is such that*

$$\frac{\partial \hat{x}_{z_i}}{\partial k} = \frac{1}{f_{x_{z_i} x_{z_i}}} \frac{\alpha_i h_{aa}}{\theta} \frac{\partial \hat{a}}{\partial k}, \quad \forall\ i = 1, \ldots, n_z, \tag{6.17}$$

$$\frac{\partial \hat{l}}{\partial k} = -\frac{h_{aa}}{\theta} \frac{\partial \hat{a}}{\partial k} \sum_{i=1}^{n_z} \frac{\alpha_i c_i}{f_{x_{z_i} x_{z_i}}} - c_1 \phi_k, \tag{6.18}$$

$$\frac{\partial \hat{z}}{\partial k} = \left[\frac{h_{aa}}{\theta} \sum_{i=1}^{n_z} \frac{\alpha_i^2}{f_{x_{z_i} x_{z_i}}} - \theta \right] \frac{\partial \hat{a}}{\partial k} + \alpha_1 \phi_k, \tag{6.19}$$

$$\frac{\partial \hat{y}}{\partial k} = \left[f_k - \phi_k f_{x_{z_1}} \right] - \phi_k \alpha_1 \frac{u_z}{u_x} + \left[-\frac{h_{aa}}{\theta} \sum_{i=1}^{n_z} \frac{\alpha_i^2}{f_{x_{z_i} x_{z_i}}} + \theta \right] \frac{u_z}{u_y} \frac{\partial \hat{a}}{\partial k}, \tag{6.20}$$

$\frac{\partial \hat{a}}{\partial k}$ solves

$$-A \left[f_k - \phi_k f_{x_{z_1}} \right] + \frac{Q \phi_k \alpha_1}{u_y} = \frac{Q}{u_y} \left[\frac{h_{aa}}{\theta} \left(u_y^2 - \sum_{i=1}^{n_z} \frac{\alpha_i^2}{f_{x_{z_i} x_{z_i}}} \right) + \theta \right] \frac{\partial \hat{a}}{\partial k} \ \textit{and,} \tag{6.21}$$

$$\frac{\partial \hat{x}_{z_i}}{\partial \tilde{g}} = \frac{\partial \hat{l}}{\partial \tilde{g}} = \frac{\partial \hat{z}}{\partial \tilde{g}} = \frac{\partial \hat{y}}{\partial \tilde{g}} = 0, \ \forall\ i = 1, \ldots, n_z. \tag{6.22}$$

Note, (6.17) follows from differentiating (6.16). Rearranging, we obtain $\frac{f_{x_{z_i} x_{z_i}}}{\alpha_i} \frac{\partial \hat{x}_{z_i}}{\partial k} = \frac{h_{aa}}{\theta} \frac{\partial \hat{a}}{\partial k}$ for all $i = 1, \ldots, n_z$, *i.e.,* as capital changes, the changes in the optimal[36] intended output-emission trade-offs should continue to be equal across all abatement strategies. Given the sign conventions we have adopted (which are indicative of the phenomenon of diminishing returns), this implies that for all $i = 1, \ldots, n_z$, $\frac{\partial \hat{x}_{z_i}}{\partial k}$ and $\frac{\partial \hat{a}}{\partial k}$ will have opposite signs.

[36] With respect to problem (6.4).

6.4. The income and substitution effects of a change in capital endowments on an economy's emission when emission is an inferior good.

In this section we will decompose the changes in solution values of intended output and emission due to a change in capital into substitution and income effects. The substitution effects arise because the optimal MRS or the trade-off between intended output and emission could change as capital level changes. The change in the MRS is attributed to the phenomenon of diminishing returns to various abatement strategies, while the signs of the income effects are governed by our assumptions that emission is an inferior good and the intended output is a normal good in consumption.

Normalizing the price of the intended output to 1, define a vector of Marshallian demands for emission and the intended output arising out of a hypothetical utility maximization exercise for any country facing $\tilde{g}$ amount of emissions from all other countries:[37]

$$\mathbf{d}(p,m) = \langle \mathbf{z}(p,m), \mathbf{y}(p,m) \rangle \equiv \text{argmax}_{y,z}\{u(y,z,\tilde{g}+z) \mid y + pz \leq m\}. \tag{6.23}$$

Define also a hypothetical expenditure minimization exercise for fixed level of utility:

$$E(p,u) \equiv \min_{y,z}\{y + pz \mid u(y,z,\tilde{g}+z) \geq u\}, \tag{6.24}$$

yielding the Hicksian demands: $\mathbf{d}^H(p,u) = \langle \mathbf{z}^H(p,u), \mathbf{y}^H(p,u) \rangle$.

Assumption (INFERIORz): Emission is an inferior good: $A \equiv u_y u_{yz} - u_z u_{yy} < 0$.

Assumption (NORMALy): Intended output is a normal good: $B \equiv u_y u_{zz} - u_z u_{yz} < 0$.

Remark 5 follows from standard comparative static exercises based on utility maximization and expenditure minimization:

Remark 5:

[37] The relative price p is negative since emission is a bad good. m is the hypothetical income of the country.

- The optimal value of the Lagrange multiplier of the utility maximization, denoted by μ, is u_y.[38]
- Strict quasi-concavity of the utility function u implies:

$$Q \equiv -\left[u_y^2 u_{zz} + u_z^2 u_{yy} - 2u_y u_z u_{yz}\right] = -\left[u_y B - u_z A\right] > 0.$$

- If Assumptions (INFERIORz) and (NORMALy) hold, then[39]

$$-A = \frac{\partial\left(-\frac{u_z}{u_y}\right)}{\partial y} > 0, \quad -B = \frac{\partial\left(-\frac{u_z}{u_y}\right)}{\partial z} > 0,$$
$$\frac{\partial \mathbf{z}}{\partial m} = \frac{\mu A}{Q} < 0, \quad \text{and} \quad \frac{\partial \mathbf{y}}{\partial m} = -\frac{\mu B}{Q} > 0.$$

- The substitution effects of a price change are given by:

$$\frac{\partial \mathbf{z}^H}{\partial p} = -\frac{\mu u_y^2}{Q} < 0 \text{ and } \frac{\partial \mathbf{y}^H}{\partial p} = \frac{\mu u_y u_z}{Q} < 0.$$

- If $E(p,u) = m$, then $\frac{\partial E}{\partial p} = \mathbf{z}^H(p,u) = \mathbf{z}(p,m)$.[40]

Lemma (SHADOWp,m) shows that for any $\langle \tilde{g}, k\rangle \in \mathbf{R}_{++} \times (0,\rho)$, there exist a pair of shadow price and income such that $\hat{y}(\tilde{g},k)$ and $\hat{z}(\tilde{g},k)$ solve the utility maximization problem (6.23). The existence of such shadow price and income follows from the first order condition (6.15) of problem (6.4).

Lemma (SHADOWp,m): *Suppose the global economy satisfies specifications (G1) to (G7) and the solution mapping Φ of (6.4) is a differentiable function. For any $\langle \tilde{g}, k\rangle \in \mathbf{R}_{++} \times (0,\rho)$ choose the shadow price and income*

$$\begin{aligned} p(\tilde{g},k) &= \frac{u_z}{u_y} = -\left[\frac{h_a}{\theta}\right] = -\left[\frac{f_{x_{z_i}} - f_l c_i}{\alpha_i}\right], \ \forall i = 1,\ldots,n_z, \ \textit{and} \\ m(\tilde{g},k) &= p(\tilde{g},k)\hat{z}(\tilde{g},k) + \hat{y}(\tilde{g},k), \end{aligned} \tag{6.25}$$

[38] The optimal value of the Lagrange multiplier of expenditure minimization is the inverse of the optimal value of the Lagrange multiplier of utility maximization.
[39] See Figures 14 and 15.
[40] This follows from Shephard's lemma and the duality between expenditure minimization and utility maximization.

where u_z, u_y, h_a, and f_z are evaluated at the optimal quantity vector $\Phi(\tilde{g}, k)$. Then

$$\begin{aligned}\mathbf{d}\left(p(\tilde{g},k), m(\tilde{g},k)\right) &\equiv \langle\ \mathbf{y}\left(p(\tilde{g},k), m(\tilde{g},k)\right),\ \mathbf{z}\left(p(\tilde{g},k), m(\tilde{g},k)\right)\ \rangle \\ &\equiv \langle\ \hat{z}(\tilde{g},k),\ \hat{y}(\tilde{g},k)\ \rangle,\end{aligned} \tag{6.26}$$

$$\frac{\partial p}{\partial k} = \frac{\partial\left(\frac{u_z}{u_y}\right)}{\partial k} = -\left[\frac{h_{aa}}{\theta}\frac{\partial \hat{a}}{\partial k}\right] = -\left[\frac{f_{x_{z_i}x_{z_i}}}{\alpha_i}\frac{\partial \hat{x}_{z_i}}{\partial k}\right],\ \forall\ i = 1, \ldots, n_z \ \text{and} \tag{6.27}$$

$$\frac{\partial m}{\partial k} = \frac{\partial p}{\partial k}\mathbf{z} + p\frac{\partial \hat{z}}{\partial k} + \frac{\partial \hat{y}}{\partial k} = -\frac{h_{aa}}{\theta}\frac{\partial \hat{a}}{\partial k}\ \mathbf{z} + f_k - \phi_k f_{x_{z_1}}. \tag{6.28}$$

Note that the change in the shadow price due to change in capital in (6.27) reflects the change in the MRS between consumption of intended output and emission, which in turn is equal to the change in the trade-off between intended production and emission generation in production. (6.25), (6.26), (6.27), and the Slutsky decomposition imply that

$$\begin{aligned}\frac{\partial \hat{z}}{\partial k} \equiv \frac{\partial \mathbf{z}}{\partial k} &= \frac{\partial \mathbf{z}}{\partial p}\frac{\partial p}{\partial k} + \frac{\partial \mathbf{z}}{\partial m}\frac{\partial m}{\partial k} \\ &= \left[\frac{\partial \mathbf{z}^H}{\partial p} - \mathbf{z}\frac{\partial \mathbf{z}}{\partial m}\right]\frac{\partial p}{\partial k} + \frac{\partial \mathbf{z}}{\partial m}\left[\mathbf{z}\frac{\partial p}{\partial k} + p\frac{\partial \hat{z}}{\partial k} + \frac{\partial \hat{y}}{\partial k}\right] \\ &= \frac{\partial \mathbf{z}^H}{\partial p}\frac{\partial p}{\partial k} + \frac{\partial \mathbf{z}}{\partial m}\left[p\frac{\partial \hat{z}}{\partial k} + \frac{\partial \hat{y}}{\partial k}\right] \ \text{and, similarly,} \\ \frac{\partial \hat{y}}{\partial k} \equiv \frac{\partial \mathbf{y}}{\partial k} &= \frac{\partial \mathbf{y}^H}{\partial p}\frac{\partial p}{\partial k} + \frac{\partial \mathbf{y}}{\partial m}\left[p\frac{\partial \hat{z}}{\partial k} + \frac{\partial \hat{y}}{\partial k}\right]\end{aligned} \tag{6.29}$$

Given $\langle \tilde{g}, k\rangle \in \mathbf{R}_{++} \times (0, \rho)$ and for $\hat{t} = \hat{z}(\tilde{g},k),\ \hat{y}(\tilde{g},k)$,[41] define the substitution effect of a change in the level of the capital input as the income compensated change in demand due to a change in prices brought about by a change in capital:

$$SE_{\hat{t}}(\tilde{g},k) \equiv \frac{\partial \mathbf{t}^H}{\partial p}\frac{\partial p}{\partial k} \tag{6.30}$$

and define the income effect of a change in the level of the capital input as the change in demand brought about by the change in *real* income brought about by a change in the capital input:

$$IE_{\hat{t}}(\tilde{g},k) \equiv \frac{\partial \mathbf{t}}{\partial m}\left[p\frac{\partial \hat{z}}{\partial k} + \frac{\partial \hat{y}}{\partial k}\right]. \tag{6.31}$$

[41] Or, equivalently, for $\mathbf{t} = \mathbf{z}(p(\tilde{g},k), m(\tilde{g},k)),\ \mathbf{y}(p(\tilde{g},k), m(\tilde{g},k))$.

so that (6.29) implies

$$\frac{\partial \hat{t}}{\partial k} \equiv \frac{\partial \hat{\mathbf{t}}}{\partial k} = SE_{\hat{t}}(\tilde{g}, k) + IE_{\hat{t}}(\tilde{g}, k). \tag{6.32}$$

The following theorem, which quantifies the income and substitution effects of a change in the level of capital, follows immediately from Remark 5 and Lemma (SHADOWp, m). If $h_{aa} = 0$ or $f_{x_{z_i} x_{z_i}} = 0$ for all $i = 1, \ldots, n_z$, *i.e.,* there are no diminishing returns to fuel inputs or the cleaning-up activity, then (6.27) implies there is no change in the shadow price (negative of the MRS) as capital changes. Hence, substitution effects are zero.

Theorem (INC&SUBST-EFFECTSz, y): *Suppose the global economy satisfies specifications (G1) to (G7), and the solution mapping Φ of (6.4) is a differentiable function. Then,*

$$\begin{aligned} SE_{\hat{z}}(\tilde{g}, k) &= \frac{\mu u_y^2}{Q} \frac{h_{aa}}{\theta} \frac{\partial \hat{a}}{\partial k}, & IE_{\hat{z}}(\tilde{g}, k) &= \frac{\mu A}{Q} \left[f_k - \phi_k f_{x_{z_1}} \right], \\ SE_{\hat{y}}(\tilde{g}, k) &= -\frac{\mu u_y u_z}{Q} \frac{h_{aa}}{\theta} \frac{\partial \hat{a}}{\partial k}, \quad \text{and} & IE_{\hat{y}}(\tilde{g}, k) &= -\frac{\mu B}{Q} \left[f_k - \phi_k f_{x_{z_1}} \right]. \end{aligned} \tag{6.33}$$

If $f_{x_z x_z} = 0$ or $h_{aa} = 0$, then $SE_{\hat{z}}(\tilde{g}, k) = 0$ and $SE_{\hat{y}}(\tilde{g}, k) = 0$.

6.5. Some special cases of the global economy satisfying (G1) to (G7).

We now consider some special cases of our model of the global economy. The first two special cases shed more light in understanding the cross-country differences in emission levels at a non-cooperative Nash equilibrium. The third special case considers the role of inter-fuel substitution in emission generation.

6.5.1. The case of zero fuel intensity of the capital input.

Consider a simple case where there is only one emission-generating input and there is no fuel cost of running capital.[42] The following theorem is a direct application of Lemma (COMP-STATICS$k, \tilde{g}$), Remark 5, and Theorems (INC&SUBST-EFFECTSz, y) and (NON-COOP-NASH k, z RELATION).

[42] In all the special cases which involve only one emission-causing input, we suppress the subscript i.

Theorem (NO-FUEL-COSTk): *Suppose specifications (G1) to (G7) of the global economy hold. If $n_z = 1$, $\phi(k) = 0$ for all k, the solution mapping Φ of (6.4) is a differentiable function, $h_{aa} > 0$, $f_{x_{z_i} x_{z_i}} < 0$ for $i = 1, \ldots, n_z$, and Assumptions (INFERIORz) and (NORMALy) hold then*

$$\begin{aligned}
&(i) \quad \frac{\partial \hat{a}}{\partial k} = \frac{-A f_k u_y f_{x_z x_z} \theta}{Q\left[h_{aa} u_y^2 f_{x_z x_z} - (h_{aa}\alpha^2 - f_{x_z x_z}\theta^2)\right]} > 0,\\
&(ii) \quad \frac{\partial \hat{x}_z}{\partial k} = \frac{1}{f_{x_z x_z}} \frac{\alpha h_{aa}}{\theta} \left[\frac{-A f_k u_y f_{x_z x_z} \theta}{Q\left[h_{aa} u_y^2 f_{x_z x_z} - (h_{aa}\alpha^2 - f_{x_z x_z}\theta^2)\right]}\right] < 0,\\
&(iii) \quad \frac{\partial \hat{z}}{\partial k} = \left[\frac{h_{aa}\alpha^2 - f_{x_z x_z}\theta^2}{\theta f_{x_z x_z}}\right] \left[\frac{-A f_k u_y f_{x_z x_z} \theta}{Q\left[h_{aa} u_y^2 f_{x_z x_z} - (h_{aa}\alpha^2 - f_{x_z x_z}\theta^2)\right]}\right] < 0,\\
&(iv) \quad \frac{\partial \hat{y}}{\partial k} = \frac{f_k \left[Q h_{aa} u_y^2 f_{x_z x_z} + u_y B \left(h_{aa}\alpha^2 - f_{x_z x_z}\theta^2\right)\right]}{Q\left[h_{aa} u_y^2 f_{x_z x_z} - (h_{aa}\alpha^2 - f_{x_z x_z}\theta^2)\right]} > 0 \text{ and,}\\
&(v) \quad SE_{\hat{z}}(\tilde{g}, k) > 0,\ IE_{\hat{z}}(\tilde{g}, k) < 0,\ SE_{\hat{y}}(\tilde{g}, k) > 0, \text{ and } IE_{\hat{y}}(\tilde{g}, k) > 0.
\end{aligned} \tag{6.34}$$

At a non-cooperative Nash equilibrium emission outcome $\langle \overset{*}{z}^1, \ldots, \overset{*}{z}^S \rangle$ of the global economy where $k^1 \le k^2 \le \ldots \le k^S$, we have $\overset{*}{z}^1 \ge \overset{*}{z}^2 \ge \ldots \ge \overset{*}{z}^S$.

Theorem (NO-FUEL-COSTk) and (6.27) imply that the MRS, $-\frac{u_z}{u_y}$, increases as capital increases. This is because as capital increases, it substitutes out the fuel input. Diminishing returns to fuel implies that the net marginal product of fuel increases. The emission intensity is constant in this model. Hence, the trade-off in production and hence consumption between intended production and emission generation rises. The substitution effect, thus, implies an increase in levels of both emissions and intended output. However, an increase in capital also implies an increase in real income by an amount f_k (see Theorem (INC&SUBST-EFFECTSz, y) and note that $\phi_k = 0$ in this case). The inferiority of the emission and normality of the intended output in consumption imply that the income effect of a change in capital on emission (resp., intended output) is negative (resp., positive). (6.34) shows that the net effect of these two effects on emission is negative, while on intended output is positive. This is illustrated with the help of preferences seen in Figure

16, where the substitution effects are indicated by a movement from point A to point B, while the income effects are indicated by a movement from point B to point C. If there are no diminishing returns, then there are no substitution effects and no changes in the MRS as capital levels change. Thus, function $\hat{z}$ is non-increasing in capital and Theorem (NON-COOP-NASH k, z RELATION) implies that at a non-cooperative Nash equilibrium, capital-rich countries generate lesser emissions and consume more intended output than capital-poor countries. The reduction in emissions as we move to countries with more and more capital is brought about by reductions in the usage of the fuel input and increases in the cleaning-up levels.

6.5.2. The case of the complementarity between capital and fuel inputs.

In this special case, we continue with the assumption of one emission-causing input. But capital is now assumed to be fuel intensive: $\phi_k > 0$ with $\phi_{kk} = 0$. In this case, it follows from Lemma (COMP-STATICS$k, \tilde{g}$) and our sign conventions that $\frac{\partial \hat{a}}{\partial k} = \frac{f_{x_z x_z}\theta[-Au_y(f_k - \phi_k f_{x_z}) + Q\phi_k \alpha]}{Q\left[h_{aa}u_y^2 f_{x_z x_z} - (h_{aa}\alpha^2 - f_{x_z x_z}\theta^2)\right]}$. Thus, $\frac{\partial \hat{a}}{\partial k}$ is greater than zero if $f_k - \phi_k f_{x_z}$ is non-negative and has an ambiguous sign otherwise. Thus, (6.27) implies that, as capital increases, the MRS, $\frac{-u_z}{u_y}$, increases when $f_k - \phi_k f_{x_z}$ is greater than zero, while the sign of the change in MRS is indeterminate for $f_k - \phi_k f_{x_z}$ less than zero. Theorem (INC&SUBST-EFFECTSz, y) thus implies that the sign of the substitution effects of a change in capital on emission and intended output levels is ambiguous when $f_k - \phi_k f_{x_z}$ is less than zero. On the other hand, this theorem also implies that, under Assumptions (INFERIORz) and (NORMALy),

$$\begin{aligned} &IE_{\hat{z}}(\tilde{g}, k) \leq 0 \text{ if and only if } f_k - \phi_k f_{x_z} \geq 0 \text{ and} \\ &IE_{\hat{y}}(\tilde{g}, k) \geq 0 \text{ if and only if } f_k - \phi_k f_{x_z} \geq 0. \end{aligned} \tag{6.35}$$

It can be shown that, when $f_k - \phi_k f_{x_z} \geq 0$, the income effects dominate the substitution effects, so that as capital increases, emission level decreases and intended output increases. Note, $f_k - \phi_k f_{x_z}$ is the increase in the real income due to a change in capital. This is the net marginal product of capital: the contribution of additional capital input to intended production *net* of the extraction cost of fuel needed to run the additional capital. Let us assume $f_{kk} > 0$, *i.e.,* there are increasing returns to capital, an assumption which may not be considered unacceptable for an input such as capital. Under our sign conventions, $\phi_k f_{x_z}$ is constant in k.[43] Hence, $f_k - \phi_k f_{x_z}$ is an increasing function of k. For low levels of capital, it could well be negative, *i.e.,* increasing returns to capital may not be strong enough yet to offset the loss in output incurred by diverting fuel inputs towards running the capital: the operation of capital equipment is hence a net drain on the economy's resources. As capital level rises, the effect of increasing returns becomes stronger and the net marginal product of capital turns positive. If (i) increasing returns to capital is true, (ii) the net marginal product of capital takes negative values for low levels of capital and positive values for high levels of capital, and (iii) the substitution effects are negligible then the function $\hat{z}$ has a maximum, and Theorem (NON-COOP-NASH k, z RELATION) implies that an environmental Kuznets curve phenomenon arises at a non-cooperative Nash equilibrium: there is an inverse U (resp., U) shape relation between capital and emission (resp., intended output) levels.

Theorem (ENV-KUZNETS-CURVE): *Suppose specifications (G1) to (G7) of the global economy hold,* $n_z = 1,\ \phi_k > 0,\ \phi_{kk} = 0,\ h_{aa} = 0$*, the solution mapping* Φ *of (6.4) is a differentiable function, and Assumptions (INFERIORz) and (NORMALy) hold. Then*

$$\begin{aligned} &(i) \quad \frac{\partial \hat{z}}{\partial k} = \frac{A u_y \left[f_k - \phi_k f_{x_z}\right]}{Q}, \\ &(ii) \quad \frac{\partial \hat{y}}{\partial k} = \frac{-B u_y \left[f_k - \phi_k f_{x_z}\right]}{Q}, \ \textit{and} \\ &(iii) \quad \frac{\partial U(\tilde{g}, k)}{\partial k} = u_y \left[f_k - \phi_k f_{x_z}\right]. \end{aligned} \tag{6.36}$$

[43] Recall, we have assumed $f_{x_z k} = 0$ and $\phi_{kk} = 0$.

If, in addition, $f_{kk} > 0$ and there exists a $\overset{*}{k}$ such that, evaluated at $\Phi(\tilde{g}, \overset{*}{k})$, $f_k - \phi_k f_{x_z} = 0$ then, at least locally in a neighbourhood around $\overset{*}{k}$, we have

$$\begin{aligned}
&(i) \quad \frac{\partial U}{\partial k} = 0 \text{ if } k = \overset{*}{k}, \quad \frac{\partial U}{\partial k} < 0 \text{ if } k < \overset{*}{k}, \quad \frac{\partial U}{\partial k} > 0 \text{ if } k > \overset{*}{k},\\
&(ii) \quad \frac{\partial \hat{y}}{\partial k} = 0 \text{ if } k = \overset{*}{k}, \quad \frac{\partial \hat{y}}{\partial k} < 0 \text{ if } k < \overset{*}{k}, \quad \frac{\partial \hat{y}}{\partial k} > 0 \text{ if } k > \overset{*}{k}, \text{ and}\\
&(iii) \quad \frac{\partial \hat{z}}{\partial k} = 0 \text{ if } k = \overset{*}{k}, \quad \frac{\partial \hat{z}}{\partial k} > 0 \text{ if } k < \overset{*}{k}, \quad \frac{\partial \hat{z}}{\partial k} < 0 \text{ if } k > \overset{*}{k}.
\end{aligned} \tag{6.37}$$

If (6.37) is true globally for all $k \in (0, \rho)$ then, at a non-cooperative Nash equilibrium emission outcome $\langle \overset{*}{z}^1, \ldots, \overset{*}{z}^S \rangle$ of the global economy where $k^1 \leq k^2 \leq \ldots \leq k^S$, (ii) of Theorem (NON-COOP-NASH k, z RELATION) holds.

The appearance of an environmental Kuznets curve at a non-cooperative Nash equilibrium in our analysis, implies that there exists a critical level $\overset{*}{k}$ of capital such that countries with capital endowment less than the critical amount cannot reap dividends from increasing returns to capital. Rather, in such countries, a large amount of resources have to be diverted away from intended production towards extraction of fuel needed for running the capital so that the net marginal product of capital is negative.

Example 2.

We now present a numerical example to demonstrate the environmental Kuznets curve. Suppose the following are true:

$$\begin{aligned}
u(y, z) &= \log(y) - \left[z^2 + z\right] + \left[\tilde{g} + z\right],\\
f(x_z, l, k) &= k^2 + l + \log(x_z + 1),\\
h(a, y) &= a + y,\\
\bar{L} = 100, \ \alpha =, \ \theta = 8, \ c &= \frac{1}{2}, \text{and } \phi_k = 2.
\end{aligned} \tag{6.38}$$

Thus, we obtain $\frac{u_z}{u_y} = -2zy$. (6.15) and Lemma (SHADOWp, m) implies: $2zy = -p(\tilde{g}, k) = \frac{1}{\theta} = \frac{1}{8}$. Thus, $\frac{\partial p}{\partial k} = 0$ and substitution effects of a change in capital are all zero. (6.16) implies $\frac{1}{x_z+1} - c = \frac{\alpha}{\theta}$, from which we derive $x_z = \hat{x}(\tilde{g}, k) = \frac{1}{7}$. $f_k - \phi_k f_{x_z} = 2k - \frac{2}{7}$, so that solving $f_k - \phi_k f_{x_z} = 0$ yields $\overset{*}{k} = \frac{7}{8}$. Solving the first-order conditions of (6.4) for functions $\hat{y}$ and $\hat{z}$, we obtain

$$\begin{aligned} \hat{z}(\tilde{g}, k) &= \frac{1}{4}\left[-1598 + 28k - 16k^2 - 16\log(\frac{8}{7}) + \sqrt{8 + \left(1598 - 28k + 16k^2 + 16\log(\frac{8}{7}\right)^2}\right] \\ \hat{y}(\tilde{g}, k) &= \frac{1}{16}\left[799 - 14k + 8k^2 + 8\log(\frac{8}{7}) + \frac{1}{2}\sqrt{8 + \left(1598 - 28k + 16k^2 + 16\log(\frac{8}{7}\right)^2}\right]. \end{aligned} \tag{6.39}$$

Figure 17 plots function $\hat{z}$, while Figure 18 plots function $\hat{y}$. The horizontal axes of both figures are reserved for capital level k. Evaluated at $\overset{*}{k} = \frac{7}{8}$, we have: $\frac{\partial \hat{y}}{\partial k} = 0$, $\frac{\partial \hat{z}}{\partial k} = 0$, $\frac{\partial^2 \hat{z}}{\partial k^2} < 0$, and $\frac{\partial^2 \hat{y}}{\partial k^2} > 0$.

6.5.3. The case of inter-fuel substitution.

We assume that there are two fuel inputs and that the use of capital is not fuel intensive. WOLOG assume that fuel input 1 is dirtier than fuel input 2, *i.e.,* $\alpha_1 > \alpha_2$. We find below that two factors determine the relative usage of the two fuels: their relative emission intensities and their relative labour costs of extraction.

We assume that what matters for intended production is the total energy generated from the use of fuel inputs. The energy generated by employing x_{z_1} and x_{z_2} amounts of fuel inputs 1 and 2 is the function: $\psi : \mathbf{R}^2_+ \longrightarrow \mathbf{R}_+$ with image $\psi(x_{z_1}, x_{z_2})$. Define the function $F : \mathbf{R}^3_+ \longrightarrow \mathbf{R}_+$ with image

$$\begin{aligned} &F(\psi(x_{z_1}, x_{z_2}), l, k) \equiv f(x_z, l, k), \quad F_\psi > 0,\ F_{\psi\psi} = 0,\ F_{\psi l} = 0\ F_{\psi k} = 0 \\ &\qquad\qquad\qquad\qquad \Longrightarrow\ f_{x_{z_i}} = F_\psi \psi_{x_{z_i}}\ \forall\ i = 1, 2. \end{aligned} \tag{6.40}$$

In particular, let's consider the case where ψ is a CES function:

$$\psi(x_{z_1}, x_{z_2}) = \left(x_{z_1}^{\frac{\sigma-1}{\sigma}} + x_{z_2}^{\frac{\sigma-1}{\sigma}}\right)^{\frac{\sigma}{\sigma-1}}, \text{ where } \sigma \neq 1 \text{ is the inter-fuel elasticity of substitution.}$$

Note that, for $i = 1,2$, $\psi_{x_{z_i}} = \frac{\left(x_{z_1}^{\frac{\sigma-1}{\sigma}} + x_{z_2}^{\frac{\sigma-1}{\sigma}}\right)^{\frac{1}{\sigma-1}}}{x_{z_i}^{\frac{1}{\sigma}}}$ and $\psi_{x_{z_i}}$ is homogeneous of degree zero in x_z. Hence, $\psi_{x_{z_1}} = \left(1 + \left(\frac{x_{z_2}}{x_{z_1}}\right)^{\frac{\sigma-1}{\sigma}}\right)^{\frac{1}{\sigma-1}}$ and $\psi_{x_{z_2}} = \left(1 + \left(\frac{x_{z_2}}{x_{z_1}}\right)^{\frac{\sigma-1}{\sigma}}\right)^{\frac{1}{\sigma-1}} \left(\frac{x_{z_2}}{x_{z_1}}\right)^{\frac{-1}{\sigma}}$.

(6.15), (6.16), and (6.40) imply that the trade-offs between emission and intended output due to changing any of the two fuel inputs are equal to the MRS in consumption:

$$\frac{F_\psi \psi_{x_{z_1}} - c_1 f_l}{\alpha_1} = \frac{F_\psi \psi_{x_{z_2}} - c_2 f_l}{\alpha_1} = -\frac{u_z}{u_y}, \tag{6.41}$$

which implies

$$F_\psi \left[\left(\frac{x_{z_2}}{x_{z_1}}\right)^{\frac{\sigma-1}{\sigma}} + 1\right]^{\frac{1}{\sigma-1}} \left[\alpha_2 - \alpha_1 \left(\frac{x_{z_2}}{x_{z_1}}\right)^{\frac{-1}{\sigma}}\right] = f_l \left[c_1\alpha_2 - c_2\alpha_1\right]. \tag{6.42}$$

(6.42) shows that, for fixed values of α_1, α_2, c_2, and σ, the optimal fuel ratio $\frac{x_{z_2}}{x_{z_1}}$ is an implicit function of c_1. It is independent of k. In fact, the following theorem shows that $\frac{x_{z_2}}{x_{z_1}}$ is an increasing function of c_1 if $\sigma > 0$. In particular, note that if $c_1 > c_2$, then the first fuel is *both* more costly to extract *and* more emission intensive than the second fuel. This implies that the right-side of (6.42) is positive. The left-side of (6.42) thus implies that $\left(\frac{x_{z_2}}{x_{z_1}}\right)^{-\frac{1}{\sigma}} < \frac{\alpha_2}{\alpha_1} < 1$. Hence, $\frac{x_{z_2}}{x_{z_1}} > 1$, *i.e.,* the second fuel input is used more intensively than the first. Since $\psi_{x_{z_1}}$ and $\psi_{x_{z_2}}$ are functions of $\frac{x_{z_2}}{x_{z_1}}$, the optimal value of which is independent of k, it follows from (6.41) and the functional form of F that $\frac{\partial \frac{u_z}{u_y}}{\partial k} = \frac{\partial p}{\partial k} = 0$, *i.e.,* there are no substitution effects of a change in capital.

Theorem (INTER-FUEL-SUBST): *Suppose specifications (G1) to (G7) of the global economy hold, $n_z = 2$, $\phi(k) = 0$ for all k, $\alpha_1 > \alpha_2$, the solution mapping Φ of (6.4) is a differentiable function, ψ is a CES function, and Assumptions (INFERIORz) and (NORMALy) hold. Then, for fixed values of α_1, α_2, c_2, and σ,*

$$\frac{\partial \frac{x_{z_2}}{x_{z_1}}}{\partial c_1} = \frac{\alpha_2 \frac{x_{z_2}}{x_{z_1}} \left(\frac{x_{z_2}}{x_{z_1}} + \left(\frac{x_{z_2}}{x_{z_1}}\right)^{\frac{1}{\sigma}}\right) \left(1 + \left(\frac{x_{z_2}}{x_{z_1}}\right)^{\frac{\sigma-1}{\sigma}}\right)^{\frac{1}{1-\sigma}} \sigma f_l}{\alpha_1 + \alpha_2 \frac{x_{z_2}}{x_{z_1}}}, \qquad \frac{\partial \frac{x_{z_2}}{x_{z_1}}}{\partial k} = 0,$$

there exists $\overset{*}{c}_1 = \left[\frac{F_\psi 2^{\frac{1}{\sigma-1}}(\alpha_2-\alpha_1)}{f_l} + c_2\alpha_1\right]\frac{1}{\alpha_2}$ *such that if* $1 \neq \sigma > 0$ *then, for any* $k > 0$,

$$\begin{aligned}\frac{x_{z_2}}{x_{z_1}}(c_1, k) &< 1 \text{ if } c_1 < \overset{*}{c}_1,\\ &= 1 \text{ if } c_1 = \overset{*}{c}_1, \text{ and}\\ &> 1 \text{ if } c_1 > \overset{*}{c}_1,\end{aligned}$$

and $SE_{\hat{y}}(\tilde{g}, k) = 0$ *and* $SE_{\hat{z}}(\tilde{g}, k) = 0$. ■

Example 3.

Figure 19 considers the case when $\sigma = 2$, $\alpha_1 = 2$, $\alpha_2 = 1$, $c_2 = 2$ and plots the relation between c_1 and the optimal value of $\frac{x_{z_2}}{x_{z_1}}$, which is given by

$$\frac{x_{z_2}}{x_{z_1}} = \frac{1}{2}\left[13 - 6c_1 + c_1^2 + (c_1 - 3)\sqrt{17 - 6c_1 + c_1^2}\right].$$

In this case, $\overset{*}{c}_1 = 2$. For $c_1 \leq \overset{*}{c}$, the dirtier fuel has a sufficient cost advantage and hence is used relatively more than the cleaner input, while if $c_1 > \overset{*}{c}$ then the dirtier fuel does not have sufficient cost advantage to warranty a greater share relative to the cleaner input in the production of energy.

7. Conclusions.

The reduced form approach that is commonly adopted in the literature to model emission-generating technologies does not distinguish between emission-causing and non-emission causing goods. It can miss out on interesting insights regarding how emissions are actually reduced when countries undertake abatement measures. Further, as pointed out in MRL, it can also result in intuitively unacceptable trade-offs in production between various goods. Here, we provide a new set of axioms to describe emission-generating technologies. Technologies that satisfy these axioms are called by-production technologies. We derive a distance function representation of by-production technologies and show that such a technology can be decomposed into a standard neo-classical intended-production technology and nature's emission generation set. Hence, the by-production approach extends neo-classical theory of production to incorporate the link in nature between emissions and emission-causing goods.

We believe that the by-production approach can yield rich results in both applied and theoretical works concerned with emission generation. As an illustrative application, we employ it to study cross-country differences in emission levels at a non-cooperative Nash equilibrium of a global economy. One of the agendas of environmental economics is to test whether economic growth is an engine of environmental consciousness. Our model (even though it is only a static one with many simplifying assumptions) of a global economy suggests that it could be. Moreover, it demonstrates a self-correcting force for the basic global environmental externality problem: environmental consciousness can increase even in the absence of any international environmental regulator as countries which were initially resource-poor begin to accumulate certain key inputs such as capital. It is not the differences in preferences for cleaner environment between rich and poor countries that explains high emission levels of developing countries. We believe all countries value clean environment: emission is perceived as an inferior good by all countries. Differences in emission levels depend on the income and substitution effects generated by differences in their productive capacities. Some countries are forced to make dirtier technological choices because they lack the right amounts of certain key highly productive inputs such as capital.

$$P(x,a) = \mathcal{P}(x,a) + (\{0\} \times \mathbf{R}_+)$$

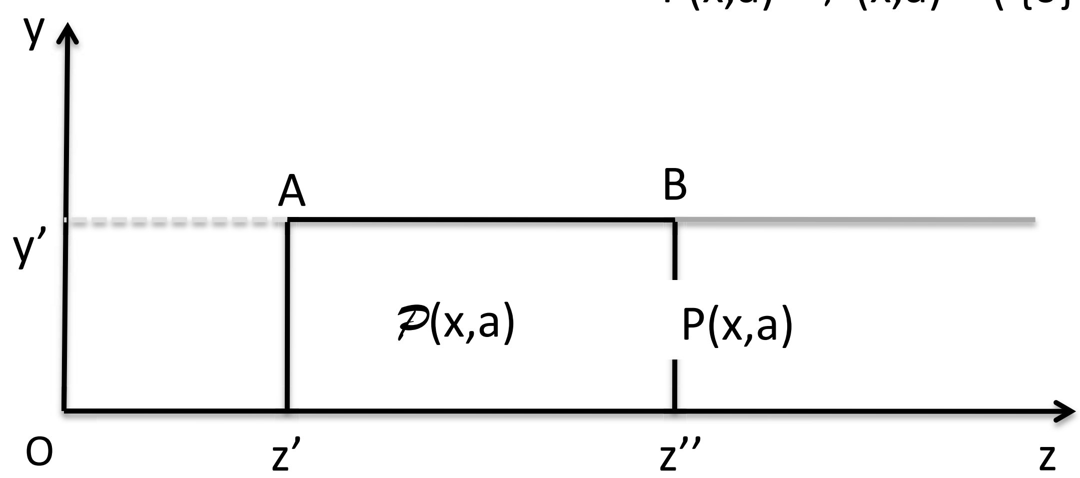


Figure 1

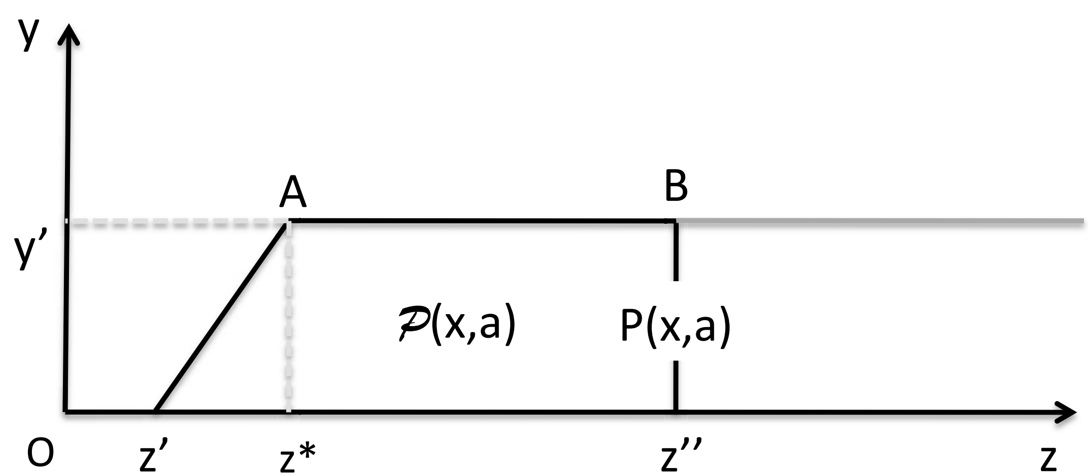
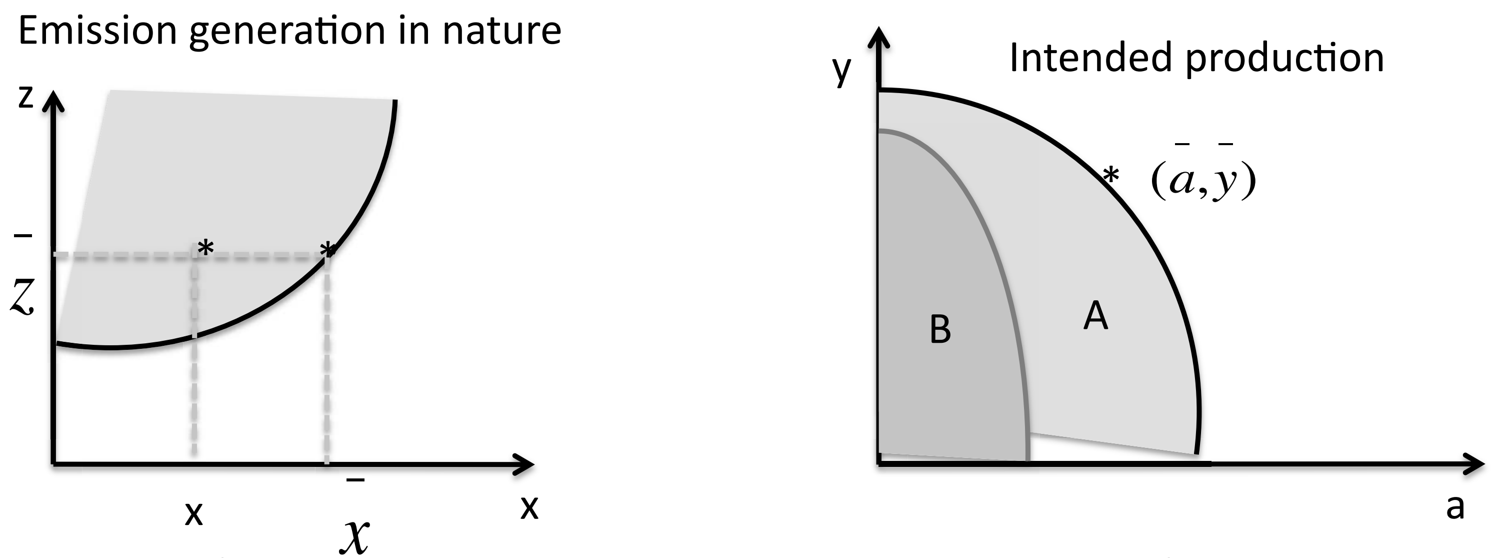
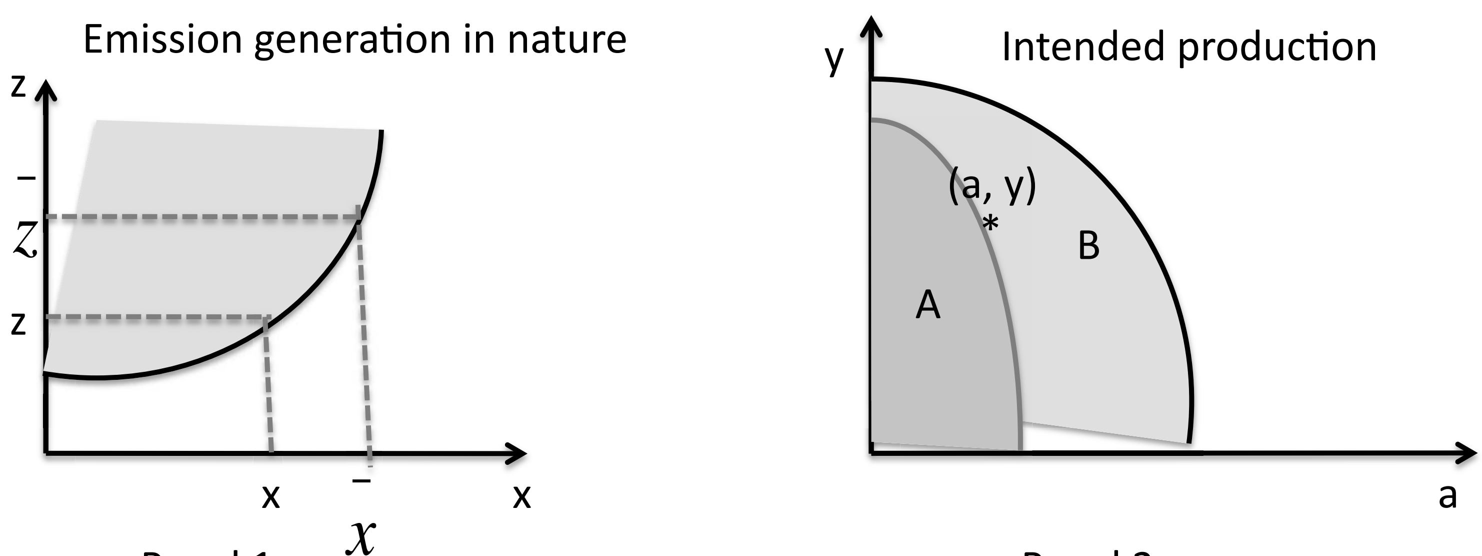


Figure 2



Panel 1 Panel 2

Figure 3



Panel 1 Panel 2

Figure 4

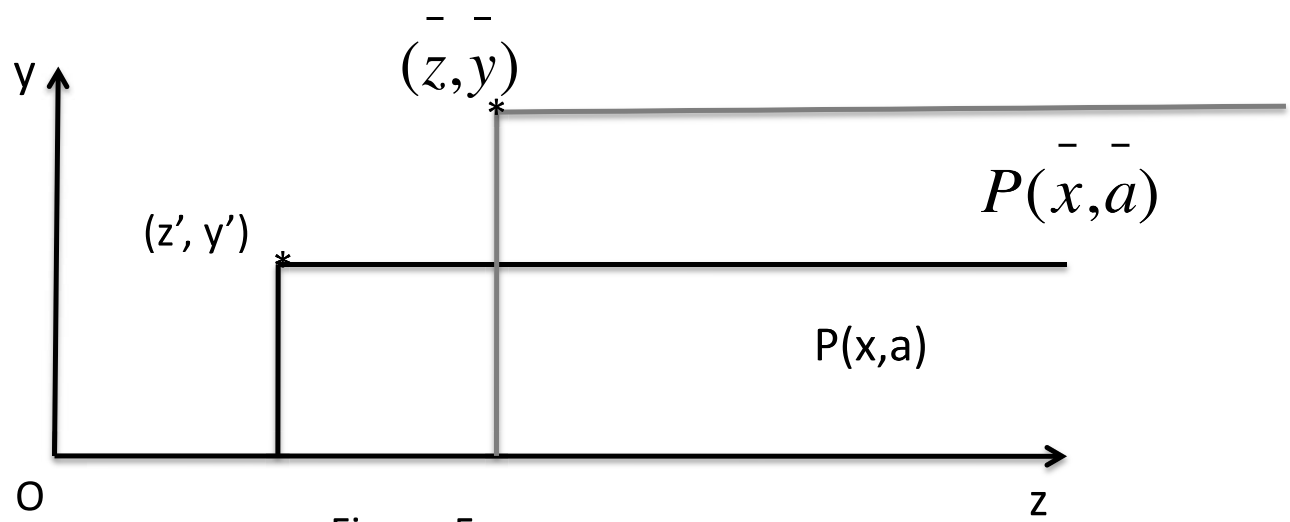


Figure 5

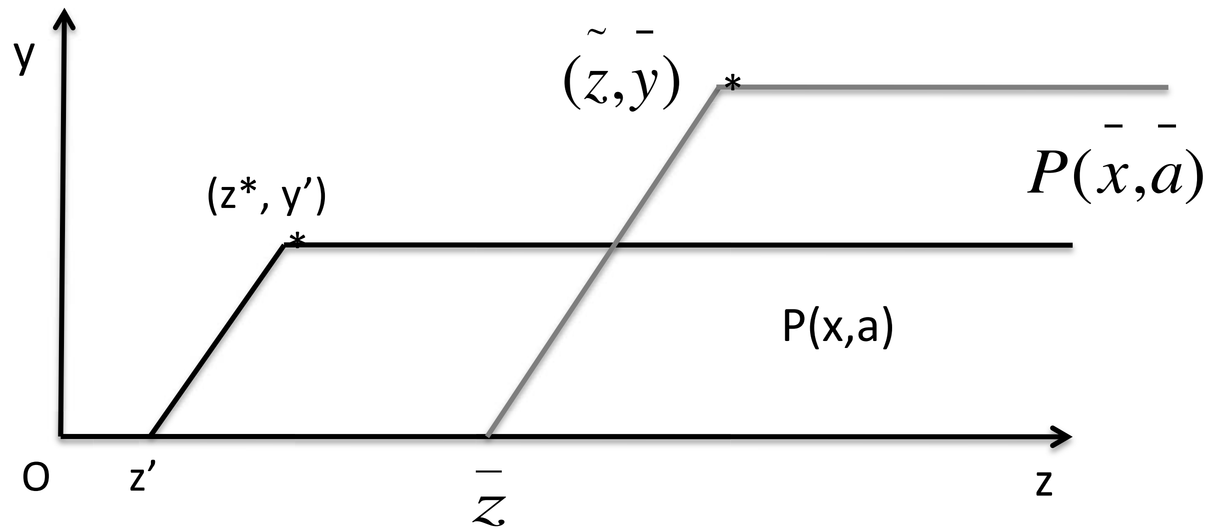


Figure 6

$$P(x,a) = \mathcal{P}(x,a) + (\{0\} \times \mathbf{R}_+)$$

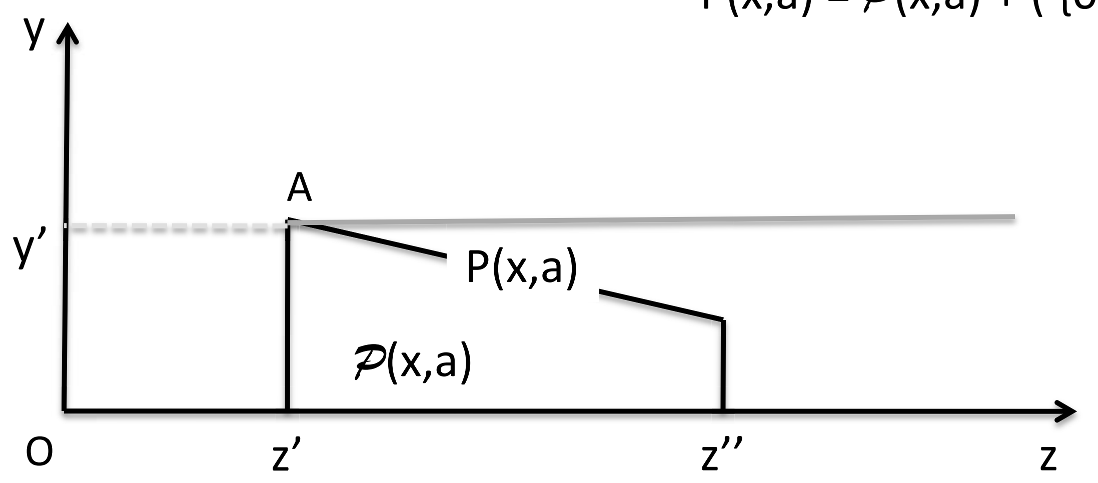


Figure 7

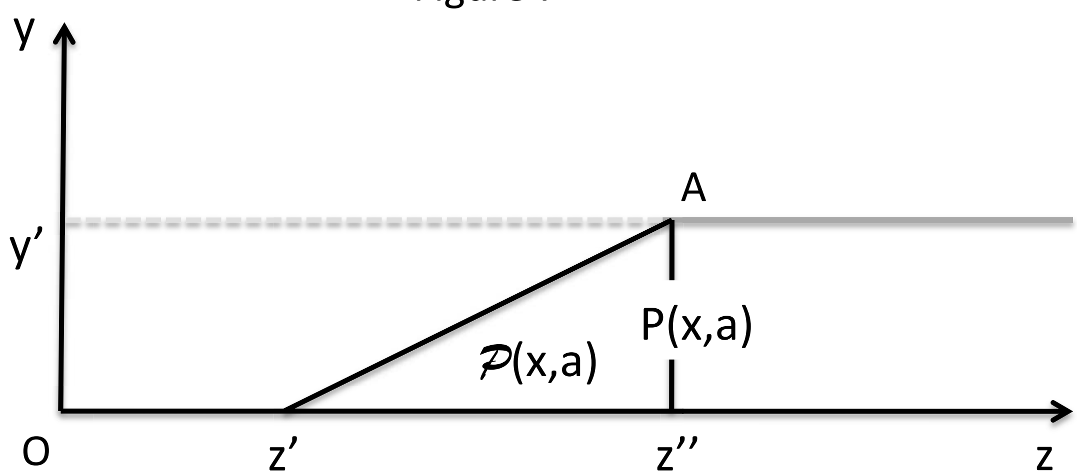


Figure 8

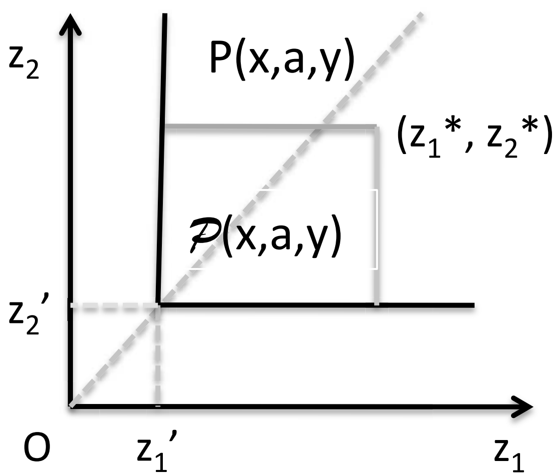


Figure 9

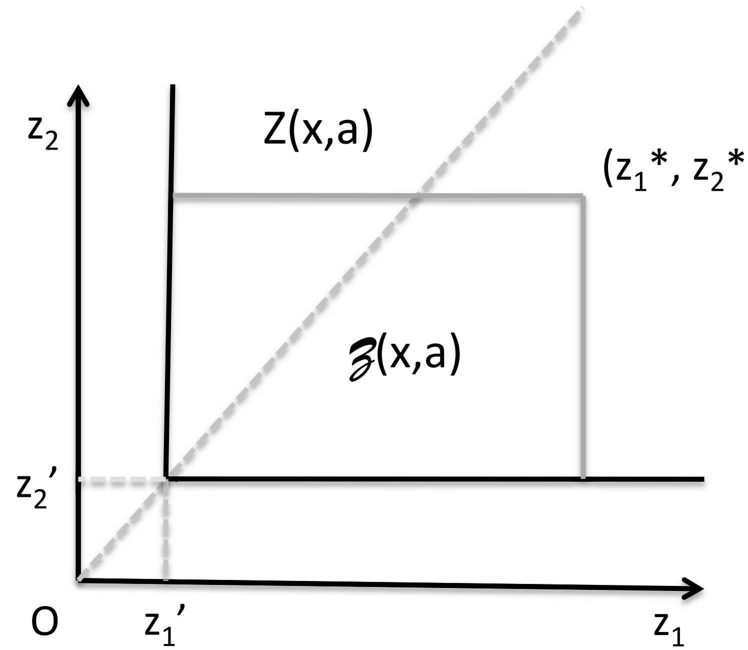


Figure 10

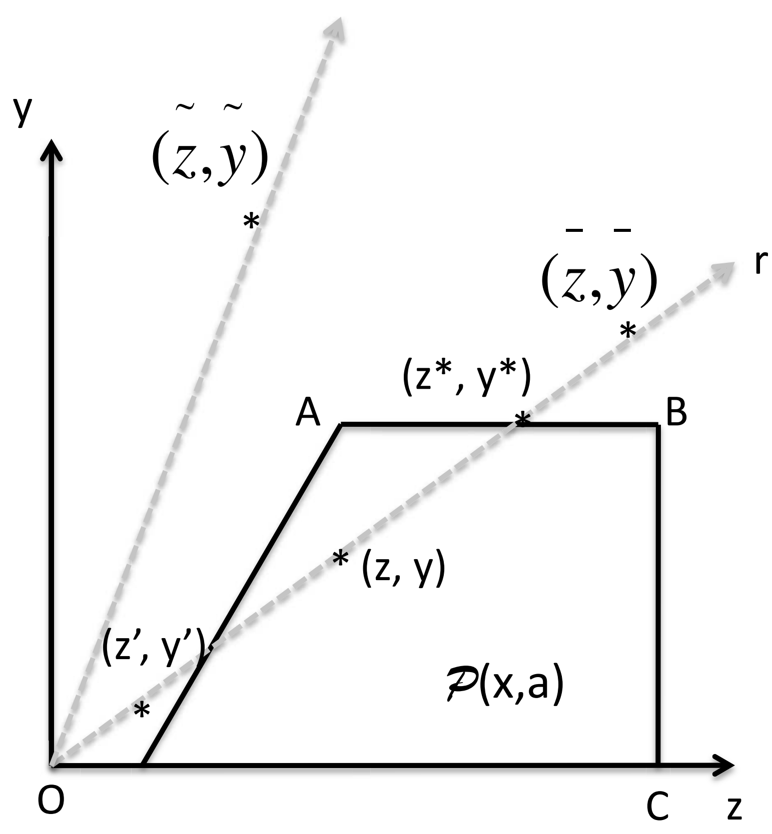


Figure 11

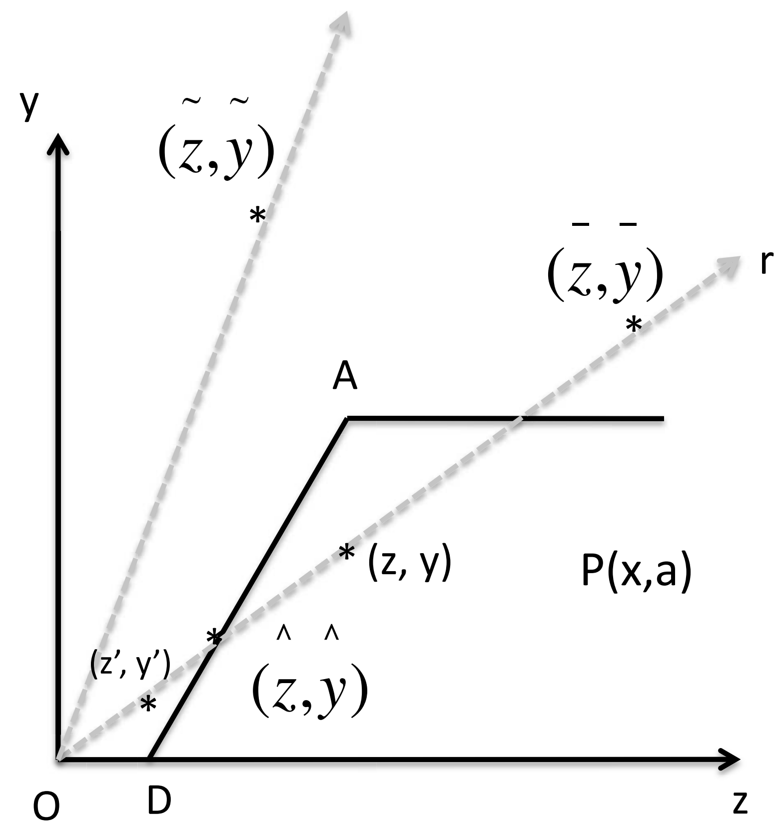


Figure 12

Figure 13

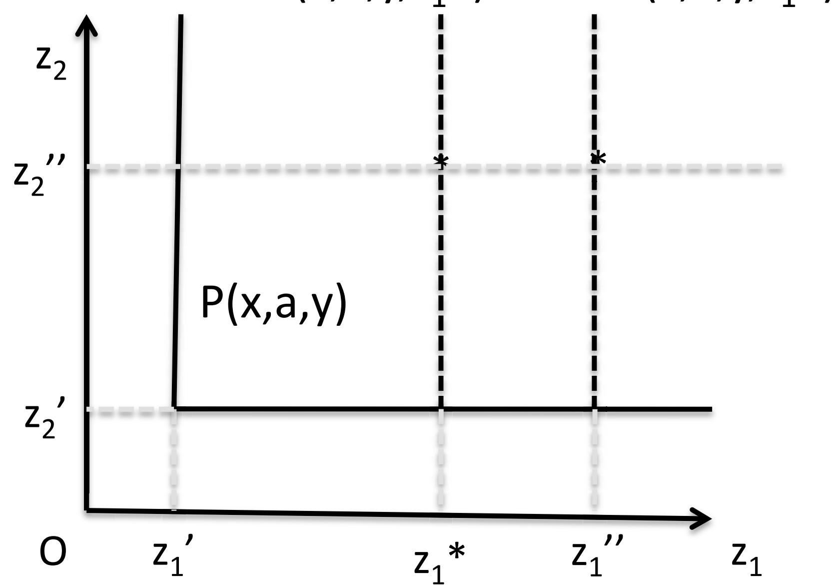


Figure 14

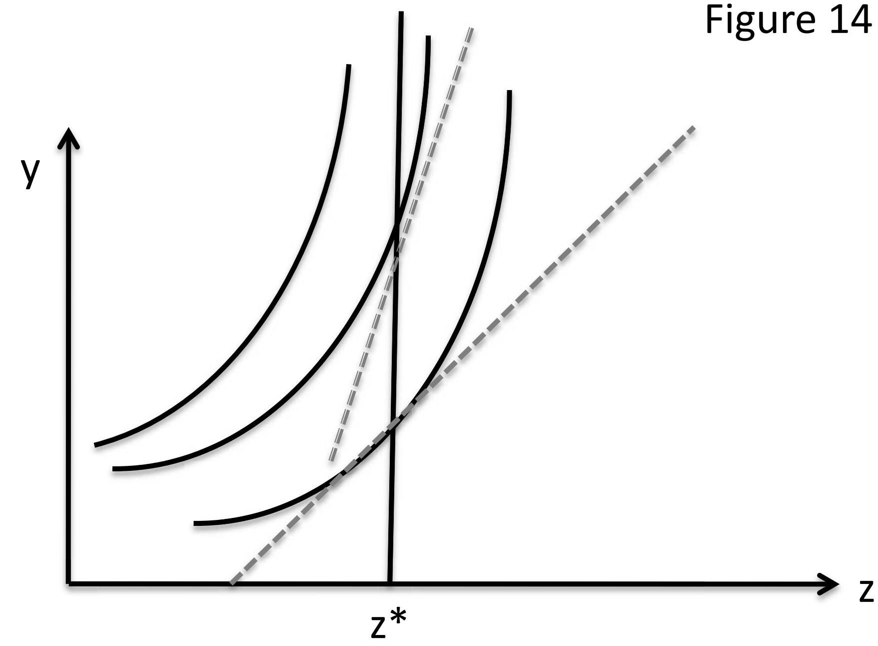


Figure 15

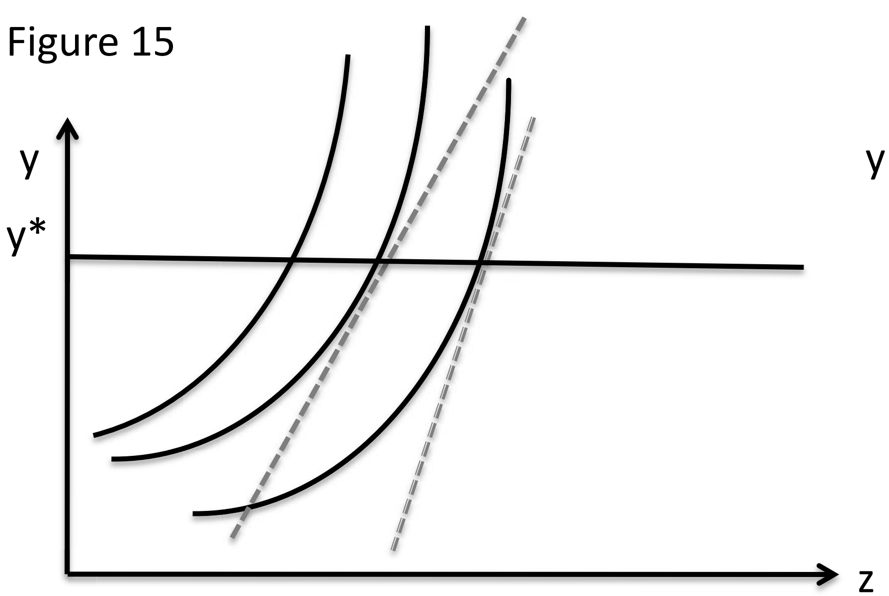


Figure 16

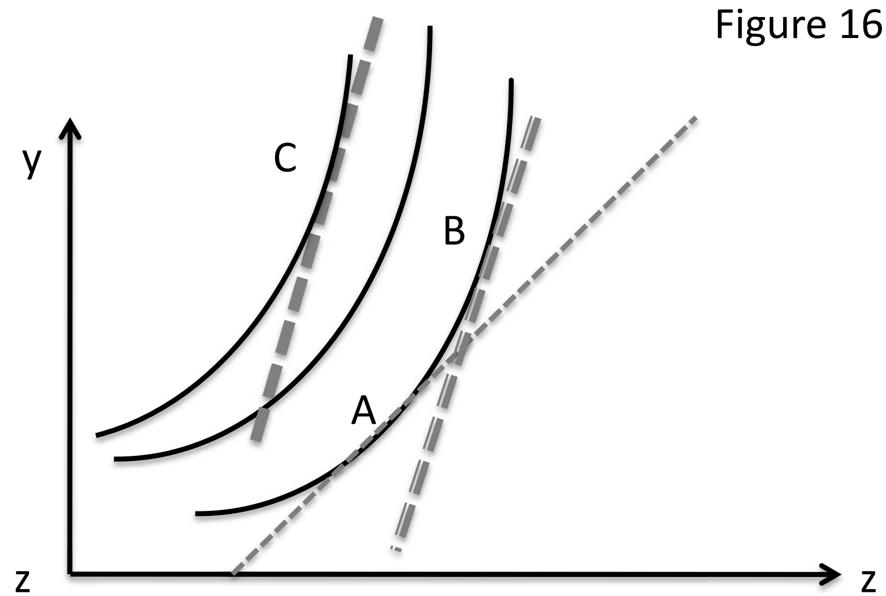


Figure 17

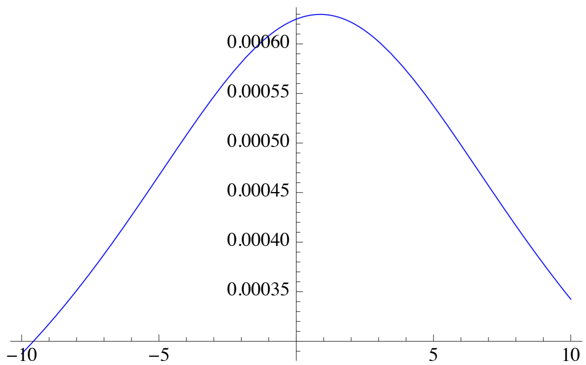


Figure 18

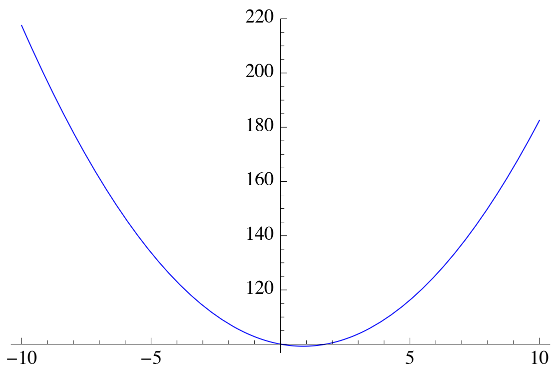
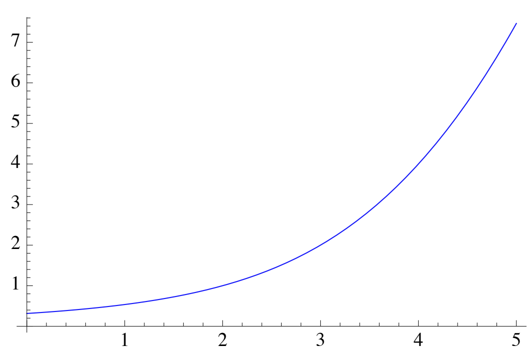


Figure 19



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APPENDIX

Proof of Theorem (BP):

(i) Let $y \in \mathcal{Y}(x,a)$. Under the maintained assumptions, there exists $z \in \mathbf{R}_+^{m'}$ and $\langle \bar{y}, \bar{z} \rangle \in \mathbf{R}_+^{m+m'}$ such that $y \in P(x,a,z)$ and $\bar{y} \in P(\bar{x},\bar{a},\bar{z})$. Choose $\hat{z}$ such that $\hat{z} \geq z$ and $\hat{z} \geq \bar{z}$. Then Remark 1 implies that $y \in P(x,a,\hat{z})$ and $\bar{y} \in P(\bar{x},\bar{a},\hat{z})$ and (CFDx_z, a) implies that $P(x,a,\hat{z}) \subset P(\bar{x},\bar{a},\hat{z})$. Hence, $y \in P(\bar{x},\bar{a},\hat{z})$. Remark 2 implies $y \in \mathcal{Y}(\bar{x},\bar{a})$.

(ii) Let $\bar{z} \in Z(\bar{x},\bar{a})$. Under the maintained assumptions, there exists $\bar{y} \in \mathbf{R}_+^m$ and $\langle y, z \rangle \in \mathbf{R}_+^{m+m'}$ such that $\bar{z} \in P(\bar{x},\bar{a},\bar{y})$ and $z \in P(x,a,y)$. Choose $\hat{y}$ such that $\hat{y} \leq y$ and $\hat{y} \leq \bar{y}$. Then Remark 1, (FDo), and (FDy_z) imply that $z \in P(x,a,\hat{y})$ and $\bar{z} \in P(\bar{x},\bar{a},\hat{y})$. (CCD$x_z, a$) implies that $P(\bar{x},\bar{a},\hat{y}) \subset P(x,a,\hat{y})$. Hence, $\bar{z} \in P(x,a,\hat{y})$. Hence, $\bar{z} \in Z(x,a)$. ■

THEOREM (IP)

Lemma (IND-IPz) is used to prove Theorem (IP). We state and prove this Lemma below before proving Theorem (IP).

Lemma (IND-IPz): *Suppose Assumptions (C), (INDz), (FDo), and (FDy_z) hold. Suppose $\langle x,a \rangle \in \Omega$ and $\langle y,z \rangle \in \mathcal{P}(x,a)$ are such that $\frac{1}{\kappa}\langle y,z \rangle \notin \mathcal{P}(x,a)$ for all $\kappa \in (0,1)$. Then, for all $z' \in \mathbf{R}_+^{m'}$ and $\kappa' \in (0,1)$, we have $\langle \frac{y}{\kappa'}, z' \rangle \notin \mathcal{P}(x,a)$.*

Proof: Suppose not. Then there exist $\kappa' \in (0,1)$ and $z' \in \mathbf{R}_+^{m'}$ such that $\langle \frac{y}{\kappa'}, z' \rangle \in \mathcal{P}(x,a)$. This implies $\frac{y}{\kappa'} \in \mathcal{P}(x,a,z')$. Choose $\hat{\kappa}$ such that $\kappa' \leq \hat{\kappa} < 1$ and $\frac{z}{\hat{\kappa}} \in \mathcal{Z}(x,a)$. Then (FD$o$) and (FD$y_z$) imply that $\frac{y}{\hat{\kappa}} \in \mathcal{P}(x,a,z')$. (Note, $\frac{y}{\hat{\kappa}} < \frac{y}{\kappa'}$.) (IND$z$) implies $\frac{y}{\hat{\kappa}} \in \mathcal{P}(x,a,\frac{z}{\hat{\kappa}})$, *i.e.*, $\langle \frac{y}{\hat{\kappa}}, \frac{z}{\hat{\kappa}} \rangle \in \mathcal{P}(x,a)$, which is a contradiction to the maintained assumptions. ■

Proof of Theorem (IP):

(i) Let $\langle x,a,y,z \rangle \in \Omega \times \mathbf{R}_+^{m+m'}$. Then from (C) and from the definition of Ω, it follows that $\mathcal{T} \neq \emptyset$ and $\mathcal{P}(x,a) \neq \emptyset$.

- Suppose $\Lambda_1(x,a,y,z) = \emptyset$. Then, by its definition, $D_1(x,a,y,z) = \infty$.
- Suppose $\Lambda_1(x,a,y,z) \neq \emptyset$. Then (BOUND$y$) and (C) imply that this set is compact. Hence, $D_1(x,a,y,z)$ is well defined and unique.

(ii) $\langle x,a,y,z \rangle \in \mathcal{T}$ implies that $1 \in \Lambda_1(x,a,y,z)$. Hence, from its definition, $D_1(x,a,y,z) \leq 1$.

- Suppose $\lambda_1 \equiv D_1(x,a,y,z) \leq 1$. Then the definition of D_1 implies that $\langle \frac{y}{\lambda_1}, \frac{z}{\lambda_1} \rangle \in \mathcal{P}(x,a)$. (FD$o$), (FD$y_z$), and $\lambda_1 \leq 1$ imply that $\langle \lambda_1 \frac{y}{\lambda_1}, \frac{z}{\lambda_1} \rangle = \langle y, \frac{z}{\lambda_1} \rangle \in \mathcal{P}(x,a)$.

(iii)

- D_1 is homogeneous of degree one in y and z:

$$\begin{aligned} D_1(x,a,\kappa y,\kappa z) &:= \inf\ \{\lambda_1 > 0 \mid \langle \frac{\kappa y}{\lambda_1}, \frac{\kappa z}{\lambda_1} \rangle \in \mathcal{P}(x,a)\} \\ &= \kappa \inf\ \{\frac{\lambda_1}{\kappa} > 0 \mid \langle \frac{y}{\frac{\lambda_1}{\kappa}}, \frac{z}{\frac{\lambda_1}{\kappa}} \rangle \in \mathcal{P}(x,a)\} \\ &= \kappa\ D_1(x,a,y,z) \end{aligned}$$

- D_1 is convex in y and z: Need to show $D_1(x,a,\alpha\bar{y}+(1-\alpha)\hat{y}, \alpha\bar{z}+(1-\alpha)\hat{z}) \le \alpha D_1(x,a,\bar{y},\bar{z}) + (1-\alpha)D_1(x,a,\hat{y},\hat{z})$ for $\alpha \in [0,1]$. Let $\bar{\lambda}_1 = D_1(x,a,\bar{y},\bar{z})$ and $\hat{\lambda}_1 = D_1(x,a,\hat{y},\hat{z})$.
  The definition of D_1 implies $\frac{1}{\bar{\lambda}}\langle \bar{y},\bar{z}\rangle \in \mathcal{P}(x,a)$ and $\frac{1}{\hat{\lambda}}\langle \hat{y},\hat{z}\rangle \in \mathcal{P}(x,a)$. Define $\overset{*}{\alpha} = \frac{\alpha\bar{\lambda}_1}{\alpha\bar{\lambda}_1+(1-\alpha)\hat{\lambda}_1}$. Then $(1-\overset{*}{\alpha}) = \frac{(1-\alpha)\hat{\lambda}_1}{\alpha\bar{\lambda}_1+(1-\alpha)\hat{\lambda}_1}$ and $\overset{*}{\alpha} \in [0,1]$. Assumption (C) implies $\langle \overset{*}{\alpha}\frac{\bar{y}}{\bar{\lambda}_1} + (1-\overset{*}{\alpha})\frac{\hat{y}}{\hat{\lambda}_1},\ \overset{*}{\alpha}\frac{\bar{z}}{\bar{\lambda}_1} + (1-\overset{*}{\alpha})\frac{\hat{z}}{\hat{\lambda}_1}\rangle \in \mathcal{P}(x,a)$.
  But $\langle \overset{*}{\alpha}\frac{\bar{y}}{\bar{\lambda}_1} + (1-\overset{*}{\alpha})\frac{\hat{y}}{\hat{\lambda}_1},\ \overset{*}{\alpha}\frac{\bar{z}}{\bar{\lambda}_1} + (1-\overset{*}{\alpha})\frac{\hat{z}}{\hat{\lambda}_1}\rangle = \langle \frac{\alpha\bar{y}+(1-\alpha)\hat{y}}{\alpha\bar{\lambda}_1+(1-\alpha)\hat{\lambda}_1},\ \frac{\alpha\bar{z}+(1-\alpha)\hat{z}}{\alpha\bar{\lambda}_1+(1-\alpha)\hat{\lambda}_1}\rangle$.
  Hence, $\alpha\bar{\lambda}_1 + (1-\alpha)\hat{\lambda}_1 \in \Lambda_1(x,a,\alpha\bar{y}+(1-\alpha)\hat{y},\alpha\bar{z}+(1-\alpha)\hat{z})$.
  Hence, $D_1(x,a,\alpha\bar{y}+(1-\alpha)\hat{y},\alpha\bar{z}+(1-\alpha)\hat{z}) \le \alpha\bar{\lambda}_1 + (1-\alpha)\hat{\lambda}_1 = \alpha D_1(x,a,\bar{y},\bar{z}) + (1-\alpha)D_1(x,a,\hat{y},\hat{z})$ for all $\alpha \in [0,1]$.
- D_1 is non-decreasing in y: Let $D_1(x,a,y,z) \equiv \lambda_1$, $\bar{y} > y$ and $D_1(x,a,\bar{y},z) \equiv \bar{\lambda}_1$. Then, the definition of D_1 implies that $\langle \frac{\bar{y}}{\bar{\lambda}_1}, \frac{z}{\bar{\lambda}_1}\rangle \in \mathcal{P}(x,a)$ and $\langle \frac{y}{\lambda_1}, \frac{z}{\lambda_1}\rangle \in \mathcal{P}(x,a)$. Clearly $\frac{y}{\bar{\lambda}_1} < \frac{\bar{y}}{\bar{\lambda}_1}$. (FD$o$) and (FD$y_z$) imply $\langle \frac{y}{\bar{\lambda}_1}, \frac{z}{\bar{\lambda}_1}\rangle \in \mathcal{P}(x,a)$. Hence, $\bar{\lambda}_1 \in \Lambda_1(x,a,y,z)$. Hence, $D_1(x,a,y,z) = \lambda_1 \le \bar{\lambda}_1 = D_1(x,a,\bar{y},z)$.
- D_1 is non-increasing in x and non-decreasing in a: Let $D_1(x,a,y,z) \equiv \lambda_1$. Let $\bar{x} \ge x$ and $\bar{a} \le a$ such that $\langle \bar{x},\bar{a}\rangle \in \Omega$ and $\langle \bar{x},\bar{a}\rangle \ne \langle x,a\rangle$. (C) and the definition of Ω imply that $\mathcal{P}(\bar{x},\bar{a}) \ne \emptyset$. Three cases are possible:
  Case 1. $\Lambda_1(\bar{x},\bar{a},y,z) = \emptyset$: Then, $D_1(\bar{x},\bar{a},y,z)$ is not well defined.
  Case 2. $\Lambda_1(\bar{x},\bar{a},y,z) \ne \emptyset$ and $\frac{z}{\lambda_1} \in \mathcal{Z}(\bar{x},\bar{a})$: Hence, there exists $y' \in \mathbf{R}^m_+$ such that $\langle y', \frac{z}{\lambda_1}\rangle \in \mathcal{P}(\bar{x},\bar{a})$. Thus, $\mathcal{P}(\bar{x},\bar{a},\frac{z}{\lambda_1}) \ne \emptyset$. Hence, (CFD$x_z,a$) implies $\mathcal{P}(x,a,\frac{z}{\lambda_1}) \subset \mathcal{P}(\bar{x},\bar{a},\frac{z}{\lambda_1})$. Hence, $\langle \frac{y}{\lambda_1}, \frac{z}{\lambda_1}\rangle \in \mathcal{P}(\bar{x},\bar{a})$. Hence, $\lambda_1 \in \Lambda_1(\bar{x},\bar{a},y,z)$. Hence, $D_1(\bar{x},\bar{a},y,z) \le \lambda_1$.
  Case 3. $\Lambda_1(\bar{x},\bar{a},y,z) \ne \emptyset$ and $\frac{z}{\lambda_1} \notin \mathcal{Z}(\bar{x},\bar{a})$: From Remark 2 and Lemma (BP), it follows that $\mathcal{Z}(\bar{x},\bar{a}) \subset Z(\bar{x},\bar{a}) \subset Z(x,a)$. Remark 1 implies that $Z(\bar{x},\bar{a}) = Z(\bar{x},\bar{a}) + \mathbf{R}^{m'}_+$. Hence, $\frac{z}{\lambda_1} < z'$ for all $z' \in Z(\bar{x},\bar{a})$. Since, $\mathcal{Z}(\bar{x},\bar{a}) \subset Z(\bar{x},\bar{a})$, $\frac{z}{\lambda_1} < z'$ for all $z' \in \mathcal{Z}(\bar{x},\bar{a})$. Hence, $\lambda_1' \le \lambda_1$ for all $\lambda_1' \in \Lambda_1(\bar{x},\bar{a},y,z)$. Hence, $D_1(\bar{x},\bar{a},y,z) \le \lambda_1$.
- D_1 is constant in z if (INDz) holds: Choose $\bar{z} \ne z$ such that $D_1(x,a,y,z) \equiv \lambda_1$ and $D_1(x,a,y,\bar{z}) \equiv \bar{\lambda}_1$ are well defined. The definition of D_1 implies that $\frac{z}{\lambda_1} \in \mathcal{P}(x,a,\frac{y}{\lambda_1})$. (IND$z$) implies $\mathcal{P}(x,a,\frac{z}{\lambda_1}) = \mathcal{P}(x,a,\frac{\bar{z}}{\lambda_1})$. Hence, $\langle \frac{y}{\lambda_1}, \frac{\bar{z}}{\lambda_1}\rangle \in \mathcal{P}(x,a)$. Hence, $\lambda_1 \in \Lambda_1(x,a,y,\bar{z})$. Hence, $\bar{\lambda}_1 \le \lambda_1$. The definition of λ_1 implies that $\frac{1}{\kappa}\langle \frac{y}{\lambda_1}, \frac{z}{\lambda_1}\rangle \notin \mathcal{P}(x,a)$

for all $\kappa \in (0,1)$. Suppose $\bar{\lambda}_1 < \lambda_1$. Then, $\frac{1}{\lambda_1}\langle y, \bar{z}\rangle = \langle \frac{\frac{y}{\bar{\lambda}_1}}{\kappa'}, \frac{\bar{z}}{\bar{\lambda}_1}\rangle \in \mathcal{P}(x,a)$, where $\kappa' \equiv \frac{\bar{\lambda}_1}{\lambda_1} \in (0,1)$. This contradicts the conclusions of Lemma (IND-IPz), which are true under the maintained assumptions. Hence, $\lambda_1 = \bar{\lambda}_1$.

- D_1 is non-decreasing in z if (DETz) holds: Choose $\bar{z} > z$ such that $D_1(x,a,y,z) \equiv \lambda_1$ and $D_1(x,a,y,\bar{z}) \equiv \bar{\lambda}_1$ are well defined. The definition of D_1 implies that $\frac{\bar{z}}{\bar{\lambda}_1} \in \mathcal{P}(x,a,\frac{y}{\bar{\lambda}_1})$. (DET$z$) implies $\mathcal{P}(x,a,\frac{\bar{z}}{\bar{\lambda}_1}) \subset \mathcal{P}(x,a,\frac{z}{\bar{\lambda}_1})$. Hence, $\langle \frac{y}{\bar{\lambda}_1}, \frac{z}{\bar{\lambda}_1}\rangle \in \mathcal{P}(x,a)$. Hence, $\bar{\lambda}_1 \in \Lambda_1(x,a,y,z)$. Hence, $\lambda_1 \leq \bar{\lambda}_1$.
- D_1 is non-increasing in z if (BENz) holds: Choose $\bar{z} > z$ such that $D_1(x,a,y,z) \equiv \lambda_1$ and $D_1(x,a,y,\bar{z}) \equiv \bar{\lambda}_1$ are well defined. The definition of D_1 implies that $\frac{z}{\lambda_1} \in \mathcal{P}(x,a,\frac{y}{\lambda_1})$. (BEN$z$) implies $\mathcal{P}(x,a,\frac{z}{\lambda_1}) \subset \mathcal{P}(x,a,\frac{\bar{z}}{\lambda_1})$. Hence, $\langle \frac{y}{\lambda_1}, \frac{\bar{z}}{\lambda_1}\rangle \in \mathcal{P}(x,a)$. Hence, $\lambda_1 \in \Lambda_1(x,a,y,\bar{z})$. Hence, $\bar{\lambda}_1 \leq \lambda_1$.

(iv) D_1 is continuous in its arguments: Proof is similar to the continuity of the output distance function in Färe and Primont [1995]. ■

THEOREM (EG)

Lemma (IND-EGo) is used to prove Theorem (EG). We state and prove this Lemma below before proving Theorem (EG).

Lemma (IND-EGo): *Suppose Assumptions (C), (FDo), and (INDo) hold. Suppose $\langle x,a\rangle \in \Omega$ and $\langle y_z, y_o, z\rangle \in P(x,a)$ are such that $\kappa\langle y_z, y_o, z\rangle \notin P(x,a)$ for all $\kappa \in [0,1)$. Then, for all $\langle x'_o, y'_o\rangle \in \mathbf{R}_+^{n_o+m_o}$ and $\kappa' \in [0,1)$, we have $\langle \kappa' y_z, y'_o, \kappa' z\rangle \notin P(x_z, x'_o, a)$.*

Proof: Suppose not. Then there exist $\kappa' \in [0,1)$ and $\langle x'_o, y'_o\rangle \in \mathbf{R}_+^{n_o+m_o}$ such that $\langle \kappa' y_z, y'_o, \kappa' z\rangle \in P(x_z, x'_o, a)$. This implies $\kappa'\langle y_z, z\rangle \in P(x_z, x'_o, a, y'_o)$. (IND$o$) implies $\kappa'\langle y_z, z\rangle \in P(x,a,y_o)$. (FD$o$) implies $\kappa'\langle y_z, y_o, z\rangle \in P(x,a)$, which is a contradiction to the maintained assumptions. ■

Proof of Theorem (EG):

(i) Let $\langle x,a,y,z\rangle \in \Omega \times \mathbf{R}_+^{m+1}$. Then from (C) and from the definition of Ω, $P(x,a) \neq \emptyset$.

- If $\Lambda_2(x,a,y,z) = \emptyset$ then $D_2(x,a,y,z) = \infty$.
- If $\Lambda_2(x,a,y,z) \neq \emptyset$, then from Remark 1, $P(x,a)$ is bounded from below. Hence, the set $\Lambda_2(x,a,y,z)$ has a lower-bound. Hence, it has a greatest lower bound. Hence, $D_2(x,a,y,z)$ is well defined.

(ii) $\langle x,a,y,z\rangle \in T \Rightarrow D_2(x,a,y,z) \leq 1$: $\langle x,a,y,z\rangle \in T$ implies $\langle y,z\rangle \in P(x,a)$. Hence, $1 \in \Lambda_2(x,a,y,z)$. Hence, $D_2(x,a,y,z) \leq 1$.

(iii)

- Proofs of linear-homogeneity and convexity of D_2 in y and z are similar to Part (iii) of Theorem (IP).
- D_2 is non-increasing in z and non-decreasing in y_z: Let $D_2(x,a,y,z) = \lambda_2$, $\bar{y}_z \leq y_z$, $\bar{z} \geq z$, and $D_2(x,a,\bar{y},\bar{z}) = \bar{\lambda}_2$. From the definition of D_2 it follows that $\langle \lambda_2 y, \lambda_2 z\rangle \in P(x,a)$. (FD$y_z$), (FD$y_o$), and Remark 1 imply that $\langle \lambda_2 \bar{y}, \lambda_2 \bar{z}\rangle \in P(x,a)$. Hence, $\lambda_2 \in \Lambda_2(x,a,\bar{y},\bar{z})$. Hence, $\bar{\lambda}_2 \leq \lambda_2$.

- D_2 is non-decreasing in x and non-increasing in a: Let $D_2(x,a,y,z) \equiv \lambda_2$. Let $\bar{x} \geq x$ and $\bar{a} \leq a$ such that $\langle \bar{x}, \bar{a} \rangle \in \Omega$ and $\langle \bar{x}, \bar{a} \rangle \neq \langle x, a \rangle$. (C) and definition of Ω imply that $P(x,a) \neq \emptyset$. Three cases are possible:
  Case 1. $\Lambda_2(x,a,y,z) = \emptyset$: Then, $D_2(x,a,y,z)$ is not well defined.
  Case 2. $\Lambda_2(x,a,y,z) \neq \emptyset$ and $\bar{\lambda}_2 y \in \mathcal{Y}(x,a)$: Hence there exists $z' \in \mathbf{R}_+$ such that $\langle \bar{\lambda}_2 y, z' \rangle \in P(x,a)$. Choose $\hat{z}$ such that $\hat{z} \geq z'$ and $\hat{z} \geq \bar{\lambda}_2 z$. Then, Remark 1 implies that $\langle \bar{\lambda}_2 y, \hat{z} \rangle \in P(x,a)$ and $\langle \bar{\lambda}_2 y, \hat{z} \rangle \in P(\bar{x}, \bar{a})$. (CCD$x_z, a$) and (FD$o$) imply that $P(\bar{x}, \bar{a}, \bar{\lambda}_2 y) \subset P(x, a, \bar{\lambda}_2 y)$. Hence, $\langle \bar{\lambda}_2 y, \bar{\lambda}_2 z \rangle \in P(x,a)$. Hence, $\bar{\lambda}_2 \in \Lambda_2(x,a,y,z)$. Hence, $D_2(x,a,y,z) \leq \bar{\lambda}_2$.
  Case 3. $\Lambda_2(x,a,y,z) \neq \emptyset$ and $\bar{\lambda}_2 y \notin \mathcal{Y}(x,a)$: From Lemma (BP), it follows that $\mathcal{Y}(x,a) \subset \mathcal{Y}(\bar{x}, \bar{a})$. (FD$o$) and (FD$y_z$) imply that $\mathcal{Y}(x,a) = \mathcal{Y}(x,a) - \mathbf{R}^m_+$. Hence, $\bar{\lambda}_2 y > y'$ for all $y' \in \mathcal{Y}(x,a)$. Hence, $\lambda_2' \leq \lambda_2$ for all $\lambda_2' \in \Lambda_2(x,a,y,z)$. Hence, $D_2(x,a,y,z) \leq \bar{\lambda}_2$.
- D_2 is constant in y_o and x_o: Proof employs Lemma (IND-EGo) and is similar to the proving that D_1 is constant in z when (INDz) holds in Part (iii) of Theorem (IP).

(iv) D_2 is continuous in its arguments: Proof is similar to the continuity of the input distance function in Färe and Primont [1995]. ■

Proof of Theorem (BP-REPR):

($\Longrightarrow$) Follows directly from Parts (ii) of Theorems (IP) and (EG).

($\Longleftarrow$) We show that $\langle x, a, y, z \rangle \notin \mathcal{T}$ implies $D_1(x,a,y,z) > 1$ or $D_2(x,a,y,z) > 1$.

- Suppose $\langle x, a, y, z \rangle \in \Omega \times \mathbf{R}^{m+m'}_+$ is such that $\lambda_2 \equiv D_2(x,a,y,z) \leq 1$ and $\langle x, a, y, z \rangle \notin \mathcal{T}$. We show that this implies $D_1(x,a,y,z) > 1$. By definition of D_2, $\langle \lambda_2 y, \lambda_2 z \rangle \in P(x,a)$.
  Case 1. $\langle \lambda_2 y, \lambda_2 z \rangle \in \mathcal{P}(x,a)$*:* We claim that if $\kappa \in \mathbf{R}_+$ is such that $\frac{1}{\kappa} \langle y, z \rangle \in \mathcal{P}(x,a)$ then $\kappa > 1$. Suppose not. Then there exists $\kappa \in (0,1)$ such that $\frac{1}{\kappa} \langle y, z \rangle \in \mathcal{P}(x,a)$. Hence, $\lambda_2 \langle y, z \rangle \ll \langle y, z \rangle \ll \frac{1}{\kappa} \langle y, z \rangle$. From (C), it follows that $\mathcal{P}(x,a)$ is convex and hence $\langle y, z \rangle$ can be written as a convex combination of $\lambda_2 \langle y, z \rangle$ and $\frac{1}{\kappa} \langle y, z \rangle$. Hence, $\langle y, z \rangle \in \mathcal{P}(x,a)$, which is a contradiction. Hence, $\kappa > 1$ for all $\kappa \in \Lambda_1(x,a,y,z)$. Hence $D_1(x,a,y,z) > 1$.
  Case 2. $\langle \lambda_2 y, \lambda_2 z \rangle \notin \mathcal{P}(x,a)$*:* Assumption $(*)$ implies $\kappa \langle \lambda_2 y, \lambda_2 z \rangle \notin \mathcal{P}(x,a)$ for all $\kappa \geq 1$. Further, if there exists $\kappa \in [0,1)$ such that $\kappa \langle \lambda_2 y, \lambda_2 z \rangle \in \mathcal{P}(x,a)$, then Remark 1 implies $\kappa \langle \lambda_2 y, \lambda_2 z \rangle \in P(x,a)$ with $\kappa \lambda_2 < \lambda_2$, which is a contradiction to $\lambda_2 = D_2(x,a,y,z)$. Hence, $\Lambda_1(x,a,y,z) = \emptyset$. Hence, $D_1(x,a,y,z) = \infty > 1$.
- Suppose $\langle x, a, y, z \rangle \in \Omega \times \mathbf{R}^{m+m'}_+$ is such that $\lambda_1 \equiv D_1(x,a,y,z) \leq 1$ and $\langle x, a, y, z \rangle \notin \mathcal{T}$. We show that this implies $D_2(x,a,y,z) > 1$. By definition of D_1, $\langle \frac{y}{\lambda_1}, \frac{z}{\lambda_1} \rangle \in \mathcal{P}(x,a)$. Hence, from our maintained assumptions, it follows that $\langle y, z \rangle \leq \frac{1}{\lambda_1} \langle y, z \rangle$. Let $\lambda_2 \equiv D_2(x,a,y,z)$.
  Case 1. $\langle \lambda_2 y, \lambda_2 z \rangle \in \mathcal{P}(x,a)$*:* We claim that $\lambda_2 > 1$. Suppose not. Then $\lambda_2 \in [0,1]$ and $\lambda_2 \langle y, z \rangle \leq \langle y, z \rangle \leq \frac{1}{\lambda_1} \langle y, z \rangle$. From (C), $\mathcal{P}(x,a)$ is convex. Hence, this implies that $\langle y, z \rangle$ is a convex combination of $\lambda_2 \langle y, z \rangle$ and $\frac{1}{\lambda_1} \langle y, z \rangle$ and hence is in $\mathcal{P}(x,a)$, which is a contradiction.

Case 2. $\langle\lambda_2 y, \lambda_2 z\rangle \notin \mathcal{P}(x,a)$: Since $\lambda_1 \leq 1$, $\langle\lambda_2 y, \lambda_2 z\rangle \in P(x,a)$, and $\frac{1}{\lambda_1}\langle y,z\rangle = \frac{1}{\lambda_2\lambda_1}\langle\lambda_2 y, \lambda_2 z\rangle \in \mathcal{P}(x,a)$, Assumption $(*)$ implies that $\lambda_2 > \frac{1}{\lambda_1\lambda_2}$. Hence, $\lambda_2^2 > \frac{1}{\lambda_1} \geq 1$. Hence, $\lambda_2 > 1$. ■

Proof of Theorem (BP-DECOMP):

- $\mathcal{T} = \mathcal{T}_1 \cap T_2$ follows from the conclusions of Theorem (BP-REPR).
- For $k = 1, \ldots, m'$, D_{2k} is well defined whenever $\langle x,a\rangle \in \Omega$, $z_{-k} \in \mathcal{Z}_{-k}(x,a,y)$, and $\Lambda_{2k}(x,a,y,z_k,z_{-k}) \equiv \{\lambda_{2k} > 0 \mid \lambda_{2k} z_k \in P(x,a,y,z_{-k})\} \neq \emptyset$. It is obvious that, under (Jz) and (INDo) with $m_z = 0$, D_{2k} is constant in z_{-k} and y whenever $\langle x,a\rangle \in \Omega$ and $z_{-k} \in \mathcal{Z}_{-k}(x,a,y)$.
- Let $\langle x,a,y,z_k,z_{-k}\rangle \in T_2$. Hence, for all $k = 1, \ldots, m$, $P(x,a,y,z_{-k}) \neq \emptyset$ and Remark 1 implies that $\mathcal{Z}_{-k}(x,a,y) \neq \emptyset$. The definition of T_2 implies, $\lambda_2 \equiv D_2(x,a,y,z) \leq 1$. The definition of D_2 implies that for all $k = 1, \ldots, m'$, $\lambda_2 z_k \in P(x,a,\lambda_2 y, \lambda_2 z_{-k})$. (J$z$) implies $\lambda_2 z_k \in P(x,a,\lambda_2 y, z_{-k})$. From Theorem (EG), (INDo) and $m_z = 0$ imply D_2 is constant in y. Hence, $\lambda_2 z_k \in P(x,a,y,z_{-k})$. Hence, the definition of D_{2k} implies $D_{2k}(x,a,y,z_k,z_{-k}) \leq \lambda_2 \leq 1$ for all $k = 1, \ldots, m'$. Hence, $\langle x,a,y,z_k,z_{-k}\rangle \in \bigcap_{k=1}^{m'} T_{2k}$.

  Let $\langle x,a,y,z_k,z_{-k}\rangle \in \bigcap_{k=1}^{m'} T_{2k}$. Then for all $k = 1, \ldots, m'$, $\lambda_{2k} \equiv D_{2k}(x,a,y,z_k,z_{-k}) \leq 1$. The definition of D_{2k} implies $\lambda_{2k} z_k \in P(x,a,y,z_{-k})$. Hence, Remark 1 implies $\frac{1}{\lambda_{2k}}\lambda_{2k} z_k = z_k \in P(x,a,y,z_{-k})$. ■

Proof of Theorem (BP-EFFICIENCY): Suppose at least one of $D_1(x,a,y,z)$ or $D_2(x,a,y,z)$ is not equal to 1.

Case 1: At least one of $D_1(x,a,y,z)$ or $D_2(x,a,y,z)$ is greater than 1: Theorem (BP-REPR) implies $\langle x,a,y,z\rangle \notin \mathcal{T}$ and hence $\langle x,a,y,z\rangle$ is not a strictly efficient point of $\mathcal{T}$.

Case 2: Suppose $\lambda_1 \equiv D_1(x,a,y,z) < 1$ and $D_2(x,a,y,z) \leq 1$: Theorem (BP-REPR) implies $\langle x,a,y,z\rangle \in \mathcal{T}$. $\frac{y}{\lambda_1} > y$ and $\frac{z}{\lambda_1} > z$. If $\mathcal{T}$ is a SBP technology, then (INDz) is true and $\mathcal{P}(x,a,z) = \mathcal{P}(x,a,\frac{z}{\lambda_1})$. If $\mathcal{T}$ is a DETBPT, then (DETz) is true and $\mathcal{P}(x,a,\frac{z}{\lambda_1}) \subset \mathcal{P}(x,a,z)$. Hence, in either case, $\langle\frac{y}{\lambda_1}, z\rangle \in \mathcal{P}(x,a)$. Thus, $\langle x,a,\frac{y}{\lambda_1}, z\rangle \in \mathcal{T}$. Hence, $\langle x,a,y,z\rangle$ is not a strictly efficient point of $\mathcal{T}$.

Case 3: Suppose $\lambda_2 \equiv D_2(x,a,y,z) < 1$ and $D_1(x,a,y,z) \leq 1$: Theorem (BP-REPR) implies $\langle x,a,y,z\rangle \in \mathcal{T}$. $\lambda_2 y < y$ and $\lambda_2 z < z$. Since $\mathcal{T}$ is a BPT and $m_z = 0$, (INDo) is true and $\mathcal{P}(x,a,y) = \mathcal{P}(x,a,\lambda_2 y)$. Hence, $\langle y, \lambda_2 z\rangle \in \mathcal{P}(x,a)$. Thus, $\langle x,a,y,\lambda_2 z\rangle \in \mathcal{T}$. Hence, $\langle x,a,y,z\rangle$ is not a strictly efficient point of $\mathcal{T}$. ■

Proof of Theorem (SOLUTION-OPT): Define the correspondence $\mathcal{C} : \mathbf{R}_{++} \times (0,\rho) \mapsto \mathbf{R}_{++}^{4+n_z}$ with image

$$\mathcal{C}(\tilde{g},k) = \big\{\langle x_z,l,a,y,z\rangle \in \mathbf{R}_{++}^{4+n_z} \bigm| D_1(x_z,l,k,a,y) \leq 1,\ D_2(x_z,l,k,a,z) \leq 1, \text{ and } \varphi(x_z,l,k) \leq \bar{L}\big\}.$$

Step 1. We show that Φ is non-empty, upper-hemicontinuous, and compact valued: this follows from Berge's theorem of the maximum once it can be shown that $\mathcal{C}$ is a non-empty, continuous, and compact-valued mapping. $\mathcal{C}$ is a non-empty and compact-valued mapping: non-emptiness follows from the definition of ρ (which defines the domain of $\mathcal{C}$) and the functional forms of D_1, D_2, and φ; convexity follows from the concavity and hence quasi-concavity of D_1 and D_2 and the linearity of φ for any fixed value of k; and compactness follows from (i) the continuity of the functions defining all constraints of (6.4) and (ii) the boundedness of the labour resource at $\bar{L}$ and the capital resource, which implies that the quantities x_z, l, y, a, and z are bounded for a given k.

We now show that $\mathcal{C}$ is also a continuous mapping. Upper-hemicontinuity of $\mathcal{C}$ follows from the continuity of D_1, D_2, and the linear function defining the third constraint of (6.4). To prove lower-hemicontinuity of $\mathcal{C}$, note that, $\mathcal{C}$ is constant in $\tilde{g}$. Consider any sequence $\{\langle \tilde{g}^\nu, k^\nu\rangle\} \to \langle \tilde{g}, \bar{k}\rangle$ and let $\langle \bar{x}_z, \bar{l}, \bar{a}, \bar{y}, \bar{z}\rangle \in \mathcal{C}(\tilde{g}, \bar{k})$. Two cases are possible:

Case 1. All of the following are true: $D_1(\bar{x}_z, \bar{l}, \bar{k}, \bar{a}, \bar{y}, \bar{z}) < 1$, $D_2(\bar{x}_z, \bar{l}, \bar{k}, \bar{a}, \bar{y}, \bar{z}) < 1$, and $\varphi(\bar{l}, \bar{x}_z, \bar{k}) < \bar{L}$: Consider the sequence $\{\langle x_z^\nu, l^\nu, a^\nu, y^\nu, z^\nu\rangle\} \to \langle \bar{x}_z, \bar{l}, \bar{a}, \bar{y}, \bar{z}\rangle$, where, for all ν, $x_z^\nu = \bar{x}_z$, $l^\nu = \bar{l}$, $a^\nu = \bar{a}$, $y^\nu = \bar{y}$, and $z^\nu = \bar{z}$. The continuity of D_1, D_2, and φ implies that there exists ν' such that for all $\nu \geq \nu'$, we have $D_1(\bar{x}_z, \bar{l}, k^\nu, \bar{a}, \bar{y}, \bar{z}) < 1$, $D_2(\bar{x}_z, \bar{l}, k^\nu, \bar{a}, \bar{y}, \bar{z}) < 1$, and $\varphi(\bar{x}_z, \bar{l}, k^\nu) < 1$. Hence, for all big enough ν, $\langle x_z^\nu, l^\nu, a^\nu, y^\nu, z^\nu\rangle \in \mathcal{C}(\tilde{g}^\nu, k^\nu)$.

Case 2. At least one of the following is true: $D_1(\bar{x}_z, \bar{l}, \bar{k}, \bar{a}, \bar{y}, \bar{z}) = 1$, $D_2(\bar{x}_z, \bar{l}, \bar{k}, \bar{a}, \bar{y}, \bar{z}) = 1$, or $\varphi(\bar{l}, \bar{x}_z, \bar{k}) = \bar{L}$: Note, (G3) implies that the function h is quasi-linear. In particular, it is linear in y, say, $h(a, y) \equiv h_y y + \bar{h}(a)$. Hence, $h(a, y) = f(x_z, l, k, y, z) \iff y = h^{-1}(a, f(x_z, l, k, z)) \iff y = f(x_z, l, k, z) - \bar{h}(a)$. Consider the sequence $\{\langle x_z^\nu, l^\nu, a^\nu, y^\nu, z^\nu\rangle\}$, where

- for all ν, $x_z^\nu = \bar{x}_z$ and $a^\nu = \bar{a}$,
- the sequence $\{z^\nu\} \to \bar{z}$ is defined such that, for all ν, $z^\nu \geq \sum_{i=1}^{n_z} \alpha_i \bar{x}_{z_i} + \alpha_1 \phi(k^\nu) - \theta\bar{a}$,[44]
- the sequence $\{l^\nu\} \to \bar{l}$ is defined such that, for all ν, $l^\nu \leq \bar{L} - c_1\phi(k^\nu) - \sum_{i=1}^{n_z} c_i \bar{x}_{z_i}$,[45]
- the sequence $\{y^\nu\} \to \bar{y}$ is defined such that, for all ν,
  $y^\nu = \tilde{y}(k^\nu) \equiv \max\{y' \mid D_1(\bar{x}_z, l^\nu, k^\nu, \bar{a}, y', z^\nu) \leq 1\} = f(\bar{x}_z, l^\nu, k^\nu, z^\nu) - \bar{h}(\bar{a})$.[46]

$\tilde{y}$ is a well-defined and continuous function. Note, $\tilde{y}(\bar{k}) = \bar{y}$. Thus, $\langle x_z^\nu, l^\nu, a^\nu, y^\nu, z^\nu\rangle \in \mathcal{C}(\tilde{g}^\nu, z^\nu)$ and $\{\langle x_z^\nu, l^\nu, a^\nu, y^\nu, z^\nu\rangle\} \to \langle \bar{x}_z, \bar{l}, \bar{a}, \bar{y}, \bar{z}\rangle$.

Step 2. We claim that $\hat{y}$ and $\hat{z}$ are continuous functions. Note, $\hat{y}$ and $\hat{z}$ are unique-valued. This follows from the fact that u is strictly quasi-concave function and $\mathcal{C}(\tilde{g}, k)$ is a convex set for all $\langle \tilde{g}, k\rangle \in \mathbf{R}_+^2$. (Convexity of $\mathcal{C}(\tilde{g}, k)$ follows from the fact that, for a fixed k, D_1 and D_2 are concave functions and φ is a linear function.) Hence, upper-hemicontinuity of Φ from Step 1 implies $\hat{y}$ and $\hat{z}$ are continuous functions. ■

Proof of Theorem (NON-COOP-NASH k, z RELATION): From the maintained assumptions of this theorem, for all $s = 1, \ldots, S-1$, we have $k^s \leq k^{s+1}$.

[44] Such a sequence exists given the continuity of the function ϕ.

[45] Such a sequence exists given the continuity of the function ϕ. Also, for big enough ν, $l^\nu > 0$ as $\bar{l} > 0$.

[46] This follows from the fact that D_1 is increasing in y and hence y^ν solves $D_1(\bar{x}_z, l^\nu, k^\nu, \bar{a}, y^\nu, z^\nu) = 1$.

- If $\hat{z}$ is non-increasing in k and constant in $\tilde{g}$, then

$$\overset{*}{z}^s \equiv \hat{z}(g^s(\overset{*}{z}^{(-s)}), k^s) \geq \hat{z}(g^s(\overset{*}{z}^{(-s)}), k^{s+1}) = \hat{z}(g^{s+1}(\overset{*}{z}^{(-(s+1))}), k^{s+1}) \equiv \overset{*}{z}^{s+1}.$$

- If $\hat{z}$ is non-decreasing in k and constant in $\tilde{g}$, then

$$\overset{*}{z}^s \equiv \hat{z}(g^s(\overset{*}{z}^{(-s)}), k^s) \leq \hat{z}(g^s(\overset{*}{z}^{(-s)}), k^{s+1}) = \hat{z}(g^{s+1}(\overset{*}{z}^{(-(s+1))}), k^{s+1}) \equiv \overset{*}{z}^{s+1}.$$

This proves (i) of this theorem. (ii) follows from (i) when we note that, for a fixed $\tilde{g}$, $\hat{z}$ is non-decreasing in the interval $(0, \overset{*}{k}]$ and non-increasing in the interval $[\overset{*}{k}, \rho)$. ∎

Proof of Lemma (COMP-STATICS$k, \tilde{g}$): For every $\langle \tilde{g}, k \rangle \in \mathbf{R}_{++} \times (0, \rho)$, Φ satisfies (6.12) to (6.16). Differentiating (6.16) with respect to k, we obtain (6.17):

$$f_{x_{z_i} x_{z_i}} \frac{\partial \hat{x}_{z_i}}{\partial k} = \frac{\alpha_i h_{aa}}{\theta} \frac{\partial \hat{a}}{\partial k} \implies \frac{\partial \hat{x}_{z_i}}{\partial k} = \frac{1}{f_{x_{z_i} x_{z_i}}} \frac{\alpha_i h_{aa}}{\theta} \frac{\partial \hat{a}}{\partial k}, \quad \forall\, i = 1, \ldots, n_z.$$

Differentiating (6.14) with respect to k and employing (6.17) we obtain (6.18). Differentiating (6.13) with respect to k and employing (6.17) we obtain (6.19). Differentiating (6.12) with respect to k and employing (6.17), (6.18), and (6.19) we obtain

$$\frac{\partial \hat{y}}{\partial k} = f_k - f_l c_1 \phi_k + f_z \alpha_1 \phi_k + \left[\frac{h_{aa}}{\theta} \sum_{i=1}^{n_z} \frac{\alpha_i}{f_{x_{z_i} x_{z_i}}} (f_{x_{z_i}} - c_i f_l + f_z \alpha_i) - \theta f_z - h_a \right] \frac{\partial \hat{a}}{\partial k}. \tag{A.1}$$

(6.20) is obtained from repeatedly employing (6.15) and (6.16) on (A.1). Differentiating (6.15), we obtain

$$\begin{aligned} -\frac{\partial \frac{u_z}{u_y}}{\partial k} &= \frac{h_{aa}}{\theta} \frac{\partial \hat{a}}{\partial k} \\ \implies -\frac{1}{u_y^2} \left[\frac{\partial \hat{y}}{\partial k} A + \frac{\partial \hat{z}}{\partial k} B \right] &= \frac{h_{aa}}{\theta} \frac{\partial \hat{a}}{\partial k}. \end{aligned} \tag{A.2}$$

(6.21) is obtained from (A.2) by employing (6.19), (6.20), (6.15), and (6.16). Note that equations (6.7) to (6.16) are all independent of $\tilde{g}$.[47] Hence, (6.22) is true. ∎

Proof of Lemma (SHADOWp, m): (6.27) follows from differentiating $p(\tilde{g}, k)$ in (6.25) with respect to k. (6.25) and (6.26) imply

$$\frac{\partial m}{\partial k} = \hat{z} \frac{\partial p}{\partial k} + p \frac{\partial \hat{z}}{\partial k} + \frac{\partial \hat{y}}{\partial k} = \mathbf{z} \frac{\partial p}{\partial k} + p \frac{\partial \hat{z}}{\partial k} + \frac{\partial \hat{y}}{\partial k}. \tag{A.3}$$

[47] The specification of u in (G7) implies u is quasi-linear. In particular, it is linear in $\tilde{g}$. Hence, u_z and u_y are independent of $\tilde{g}$.

Employing (6.27), (6.19), (6.20), (6.15), and (6.16), we obtain

$$\begin{aligned}\frac{\partial m}{\partial k} &= -\frac{h_{aa}}{\theta}\frac{\partial \hat{a}}{\partial k}\,\mathbf{z} + \left[\frac{u_z}{u_x}\frac{h_{aa}}{\theta}\sum_{i=1}^{n_z}\frac{\alpha_i^2}{f_{x_{z_i}x_{z_i}}} - \frac{u_z}{u_x}\theta\right]\frac{\partial \hat{a}}{\partial k} + \frac{u_z}{u_x}\alpha_1\phi_k + f_k - f_l c_1\phi_k + f_z\alpha_1\phi_k \\ &\quad + \left[\frac{h_{aa}}{\theta}\sum_{i=1}^{n_z}\frac{\alpha_i}{f_{x_{z_i}x_{z_i}}}(f_{x_{z_i}} - c_i f_l + f_z\alpha_i) - \theta f_z - h_a\right]\frac{\partial \hat{a}}{\partial k} \\ &= -\frac{h_{aa}}{\theta}\frac{\partial \hat{a}}{\partial k}\,\mathbf{z} + \frac{u_z}{u_x}\alpha_1\phi_k + f_k - f_l c_1\phi_k + f_z\alpha_1\phi_k \\ &= -\frac{h_{aa}}{\theta}\frac{\partial \hat{a}}{\partial k}\,\mathbf{z} + f_k - \phi_k f_{x_{z_1}}.\ \blacksquare\end{aligned} \tag{A.4}$$

Proof of Theorem (ENV-KUZNETS-CURVE) Since $h_{aa} = 0$, from Lemma (Shadowp, m) it follows that $\frac{\partial p(k)}{\partial k} = 0$ and $\frac{\partial m(k)}{\partial k} = f_k - \phi_k f_{x_z}$. The substitution effects of a change in k are zero, hence (ii) and (iii) of (6.37) reflect only the income effects of a change in k. From the envelope theorem, $\frac{\partial U}{\partial k} = \lambda f_k - \delta\alpha\phi_k - \gamma c\phi_k$. Employing (6.7) and (6.11) to this, we obtain (iii) of (6.36). From (6.17) it follows that $\frac{\partial \hat{x}}{\partial k} = 0$. Hence, at $\overset{*}{k}$, $\frac{\partial^2 U}{\partial k^2} = u_y f_{kk} > 0$.Thus, $\overset{*}{k}$ is a (local) minimum of the function U. Similarly, $\overset{*}{k}$ is a (local) minimum (resp., maximum) of the function $\hat{y}$ (resp., $\hat{z}$). $\blacksquare$