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An OLG Model of Endogenous Growth and Ageing*

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Abstract

To study fully endogenous ageing alongside growth and their implications, this paper sets up an OLG economy in which the life expectancy of agents is endogenous. Agents are bearers of children, investors in education and producers and consumers of output. Retirement decision in the last period of life is also endogenous and, when retired, agents receive a pension that is financed by a PAYG system. The model features a unique asymptotically stable steady state. Accordingly it supports a plethora of short and medium-run dynamic paths to a uniquely defined long-run equilibrium depending on the functional forms, parameters and initial conditions. The model is then calibrated and some comparative static experiments on the effect of policy variables changes are conducted.

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1 Introduction

Significant demographic changes are predicted to take place in all industrialised countries over the next 25 years. Some of the most striking projected changes are that the ratios of people aged 60 and above to 20-59 year old are predicted to rise significantly in all countries; the 0-19 year old ratios are generally predicted to fall and, generally, at a rather more modest rate; and there are marked differences across countries especially in the size of the predicted rises in the 60-and-above dependency ratios. (See for example Population Trends, ONS, Winter 1997.)

Given these facts, in recent years there has been an increasing interest in the economic literature into the causes and implications of an ageing population. However, there has been relatively little research into the macroeconomic effects of such changes. The research that has been done suggests that the "broader macroeconomic consequences [of an ageing population] may be great and may in fact overwhelm the consequences for social security alone" (Johnson and Falkingham (1992, p.153)). However, while the empirical literature is fast growing, the theoretical underpinning of an economic analysis of population ageing is lacking behind. In fact, the demo-economic theory is mainly concerned with the early demographic transition. In particular, numerous studies concentrate on the fall of fertility as per capita income increases. Yet there is a clear lack of theoretical research on the role of longevity.

In the neo-classical growth theory, population was treated as an exogenous variable that simply grew at a constant, autonomous rate. Accordingly this theory was silent on the relationship between economic and demographic outcomes and could not address the evidence on the variations in fertility and mortality rates across different economies or in the same economy over time. To explain such evidence, it was necessary to depart from neo-classical theory and to develop models in which population was determined endogenously. The seminal paper by Becker (1960) was the first attempt to fill the gap by reformulating the classical theory of population in modern economic terms. The radical contribution of that paper was the development of a model where parents derive utility from their own consumption as well as from the quantity and quality of children, who are treated as durable goods. The analysis established the micro-foundations of a theory of population from which an extensive literature on endogenous population has since developed. Some of the early topics to be considered were the effect of social security on fertility (Becker and Barro, 1988; Cigno and Rosati, 1992), the interaction between the quantity and quality of children (Becker and Lewis, 1973) and the relationship between capital markets and intergenerational transfers (Nerlove, Razin and Sadka, 1991).

The development of endogenous growth theory (initiated by Lucas 1988 and Romer 1986) offered a framework for gaining further insights into the relationship between population change and economic activity. The major challenge has been to explain demographic transition in fully articulated dynamic general equilibrium models in which the development of the economy may be unbounded. One of the first models in this genre is that of Becker, Murphy and Tamura (1990) who consider a dynastic model of intergenerational transfers where parents derive utility from the number of children and the utility level that each child receives. Human capital accumulation is the key to growth in this model and is determined endogenously, together with the number of children. The model produces two stable equilibria. One of these is a development trap where fertility is high, there is no investment in the human capital of one's children and the rate of growth is zero. The other features low fertility, positive investment in human capital and positive growth. Which equilibrium transpires depends crucially on initial conditions. The model captures the empirical evidence on fertility and growth, though the transition from one equilibrium to the other cannot be explained. Another early contribution is that of Ehrlich and Lui (1991) who consider a different model in which parents invest in their children's human capital to achieve both old-age support and emotional gratification, where material support from children is determined through self-enforcing implicit contracts. One implication of this model is that fertility is at a corner, with each parent choosing the minimum number of children in equilibrium.

These and other contributions (like Galor and Weil (2000) and Tamura (1996)) provide an explanation only for some aspects of the demographic transition. In particular they do not attempt to study ageing in a model where demographic - including variations in longevity - and economic outcomes are determined *jointly* within an economy. In other words, the received literature does not study *fully* endogenous ageing and its implications for growth, retirement and savings decisions, and the tax burden. In fact, where longevity has been taken into account (for example by Ehrlich and Lui (1991)), it is still treated as exogenous. Our paper is an attempt to fill this gap.

To study fully endogenous ageing alongside growth and their implications - like in the paper by Becker, Murphy and Tamura (1990) - we use an OLG framework to investigate the relationship between human capital accumulation, fertility and economic growth. The novelty in our paper, however, is the endogenous treatment of longevity. We believe that such a treatment is important because in industrialised countries the ageing of the population is as much due to increased longevity as reduced fertility. Therefore, a necessary ingredient of a fully-fledged model of ageing is endogenous longevity as well as endogenous fertility.[1]

[1] To the best of our knowledge, only two recent and unpublished papers by Lagerlöf (2000) and Blackburn and Cipriani (2000) treat longevity as endogenous. Nevertheless, these papers do not focus on endogenous ageing. The first paper is concerned with explaining the transition from a Malthusian Regime to a Modern Growth Regime. Therefore it does not consider a pension scheme and the resulting tax burden or the retirement decision. Blackburn and Cipriani (2000) focus on the effect of increasing longevity on the timing of childbearing and on educational investment. Thus, they again do not consider pension and other related issues. Also, both papers do not feature savings. Finally, both papers make specific assumptions on technologies and preferences while our model is derived under very general assumptions.

We consider an overlapping generations economy in which the life expectancy of agents extends probabilistically to three periods. Agents are bearers of children, investors in education and producers and consumers of output. Child-bearing is costly. As in Blackburn and Cipriani (2000), educational investment when young is the means of accumulating human capital which raises the future productivity of labour. This type of human capital function is based on Uzawa (1965) and Lucas (1988). Retirement decision in the last period of life is endogenous and when retired, agents receive a pension which is financed by a PAYG system. The model is firstly derived under general forms for the utility and the human capital accumulation functions.

We show that the model features a unique asymptotically stable steady state. Accordingly our model supports a plethora of short and medium-run dynamic paths to a uniquely defined long-run equilibrium depending on the functional forms, parameters and initial conditions. This result comes in striking contrast to the existing ones, for example to the result of Becker et al. (1990). In more detail, Becker et al. (1990) find multiple long-run equilibria, whilst we find a unique steady state. In Becker et al. (1990) one steady state is unstable and the rest two are locally stable, whilst our long-run equilibrium is a sink.

Under specific assumptions for preferences and technology we show by means of simulations that our model is capable of generating testable time series which accord with the empirical evidence on the relationship between demography and development. In particular, the fertility rate is initially decreasing and then it increases as in the baby boom to fall again later. Longevity is always increasing while the savings rate, the retirement age and the tax rate are non monotonic. The final part of the paper considers some comparative static experiments on the effect of policy variables changes and shows the effects of a child rearing subsidy and of a less generous pension scheme on ageing and growth.

2 The model

Time is discrete and the superscript t denotes a variable which refers to a generation born at time t. There is an endogenous population of reproductive agents belonging to overlapping generations with finite but uncertain lifetimes. Each agent matures safely from childhood to adulthood and has a probability of surviving to old age. In the first period of life an agent is raised by her parent and decides how much time to invest in education or to devote to leisure. When adult he becomes active as a bearer of children and a producer and consumer of output. If she survives to old age, an agent has to decide when to retire and enjoy a pension financed with a PAYG system. The endowment of time in each period is normalised to one. All agents have identical preferences and technologies, and are aware of their life expectancies.

The expected lifetime utility of an agent of generation t is given by

$$W^t = U(1-e^t) + \delta\phi(c_a^t) + \beta\phi(n^t) + \pi^t(\varphi\phi(c_o^t) + \gamma U(1-l^t)) \tag{1}$$

where e^t denotes time spent in education when young and l^t denotes the fraction of time spent on working in old age. Moreover, c_a^t and c_o^t denote consumption when adult and old respectively, n^t is the number of children and π^t is the probability of surviving to the third period. As in other papers in this literature, parents derive utility from the production of offspring and children are treated as consumption goods, yielding utility to their parents only during the period in which they are born. For simplicity, agents are assumed to enjoy leisure only when young and when old. The functions U and ϕ are strictly increasing and concave as usual. Finally, β, φ and γ are strictly positive scalars, and $\delta \in (0,1)$ denotes the discount factor.[2]

Each agent enters her second period of life with an amount of human capital which is partly inherited from her parent and partly the result of his own educational effort according to the following function

$$h^t = H(h^{t-1}, e^t). \tag{2}$$

As an adult, each agent spends all her time endowment on work. The consumption level of an adult/young agent is

$$c_a^t = (1 - qn^t + \tau^t - s^t)wh^t \equiv L_a^t wh^t, \tag{3}$$

where w is the fixed wage per unit of effective labour,[3] q is the (constant) cost of raising children net of a fixed subsidy on child care a as a proportion of the effective wage wh^t. In addition, s^t is the saving rate and τ^t is the net transfer to generation t in period *t+1,* as proportions of the effective wage. L_a^t can be thought of as the consumption generating time when young net of all possible (net) outflows measured in time-equivalent units.

If an agent survives to the third period, then she allocates her time between working and leisure (retirement). When retired she receives a pension which is a proportion p of her effective wage prior to retirement. Denoting with R the fixed, exogenous, rate of interest determined at the world level[4], consumption in old age is

$$c_o^t = \left[l^t + p\left(1 - l^t\right) + s^t R\right] wh^t \equiv L_o^t wh^t. \tag{4}$$

L_o^t is the consumption generating time when old plus any possible inflow measured in time-equivalent units.

The net transfer received by adults of generation t is determined by a balanced budget condition for financing pensions and child care subsidies, where

[2]Note that the scalars β, φ and γ incorporate implicitly the discount factor δ, since they can be thought of being such that $\beta = \delta\hat{\beta}$, $\varphi = \delta^2\hat{\varphi}$ and $\gamma = \delta^2\hat{\gamma}$ where $\hat{\beta}, \hat{\varphi}$ and $\hat{\gamma}$ are strictly positive scalars.

[3]For our assumption that w is exogenously given, see next footnote.

[4]The usual small open economy assumption allows us to consider both R and w as exogenous variables determined at the world level.

the income from savings as an adult of agents of generation $t-1$ that do not enter the third period (i.e. unintended bequests) are left to/taxed at a 100% rate by the Government for simplicity. That is,

$$\tau^{t}=\frac{h^{t-1}\left[s^{t-1}R\left(1-\pi^{t-1}\right)-p\left(1-l^{t-1}\right)\pi^{t-1}\right]}{h^{t}n^{t-1}}-an^{t}. \tag{5}$$

The model is completed by specifying one important feature which is the endogenous determination of the survival probability, π. By endogenising longevity the paper distinguishes itself from most of the existing literature. Like in Blackburn and Cipriani (2001) and in Lagerlöf (2000) we assume that longevity depends on human capital. In particular we assume that the longevity of generation t depends on the average human capital level of that generation to reflect the fact that better educated individuals are more likely to adopt healthy lifestyles:

$$\pi^{t}=\pi\left(\overline{h}^{t}\right) \tag{6}$$

where $\overline{h}^{t}$ is the average level of human capital of generation t, $\pi'(.)>0$, $\pi(0)=\underline{\pi}$ and $\lim_{\overline{h}->\infty}\pi\left(\overline{h}^{t}\right)=\overline{\pi}\leq 1$.[5]

The typical agent born in period t is faced with the problem of maximising (1) with respect to e^t, c_a^t, c_o^t, n^t, l^t subject to (2)-(4) and (6), taking as given τ^t, p, $\overline{h}^t$, h^{t-1}, q, w and R. Note that in equilibrium $\overline{h}^t=h^t$ and that the equilibrium paths for e^t, c_a^t, c_o^t, n^t, s^t, l^t, h^t, π^t, τ^t follow from the solution to the above problem, (2)-(6) and $\overline{h}^t=h^t$ as a function of p, a, q, h^{t-1}, n^{t-1}, s^{t-1}, l^{t-1}, π^{t-1}, w and R.

Finally, note that the demographic composition of the population is endogenously determined in our model. In more detail at any instant t the number of adults N^t is $N^{t-1}n^{t-1}$, the number of children is $N^{t-1}n^{t-1}n^t$, and the number of old agents is $N^{t-1}\pi^{t-1}$. Given these we have that the ratio of old agents to adults is π^{t-1}/n^{t-1}, whilst the ratio of young agents to adults is n^t. Note that the sum of these two ratios determine the dependency ratio of our economy, or in other words the ratio of young and old agents and to adults. Moreover, we have the following ageing indices: the ratio of old to young agents is $\frac{\pi^{t-1}}{n^{t-1}n^t}$ whilst the ratio of old agents to the total population is $\frac{\pi^{t-1}}{\pi^{t-1}+n^{t-1}(1+n^t)}$. We now turn to the characterisation of equilibria.

3 Equilibrium Characterisation

Assuming for simplicity an interior solution the first order conditions (FOCs hereafter) of the problem faced by the typical agent t with respect to e^t, n^t, s^t and l^t are:

$$[\delta\phi'(c_a^t)L_a^t+\pi^t\varphi\phi'(c_o^t)L_o^t]H_2^t=U'(1-e^t) \tag{7}$$

[5]Primes denote first derivatives and double primes denote second derivatives.

$$\delta\phi'(c_a^t)H^t q = \beta\phi'(n^t) \tag{8}$$

$$\pi^t\varphi\phi'(c_o^t)R = \delta\phi'(c_a^t) \tag{9}$$

$$\varphi\phi'(c_o^t)H^t(1-p) = \gamma U'(1-l^t), \tag{10}$$

respectively. In the above $H^t \equiv H(h^{t-1}, e^t)$ and $H_i^t > 0$ denotes the first partial derivative of $H(h^{t-1}, e^t)$ with respect to its i^{th} argument. In what follows we will also denote as H_{ij}^t the first partial derivative of H_i^t with respect to its j^{th} argument. We assume hereafter that $H_{21}^t > 0$ and $H_{22}^t < 0$.

Before we proceed into the characterisation of equilibria we briefly discuss the incentives faced by the typical agent t when she decides upon e^t, n^t, s^t and l^t for any given w, R, τ^t, p, $\overline{h}^t$, h^{t-1} and q, as these are implicitly determined by (7)-(10). Using (2)-(4), conditions (7)-(10) can be rewritten as

$$D_e(e^t, L_a^t, L_o^t, \pi^t, h^{t-1}, w) = 0 \tag{11}$$

$$D_n(e^t, n^t, L_a^t, q, h^{t-1}, w) = 0 \tag{12}$$

$$D_s(e^t, L_a^t, L_o^t, \pi^t, h^{t-1}, w, R) = 0 \tag{13}$$

$$D_l(e^t, l^t, L_o^t, p, h^{t-1}, w) = 0 \tag{14}$$

where D_i denotes the welfare gain from a marginal increase in i - given w, R, τ^t, p, $\overline{h}^t$, h^{t-1}, q - on the part of the typical agent t as a function of e^t, n^t, l^t, L_a^t, L_o^t, π^t, h^{t-1}, p, q, w and R. The definitions for L_a^t and L_o^t can also be re-written as

$$L_a^t - (1 - q^t n^t + \tau^t - s^t) = 0 \tag{15}$$

and

$$L_o^t - \left[l^t + p^t\left(1 - l^t\right) + s^t R\right] = 0. \tag{16}$$

The above system gives implicitly the decisions on e^t, n^t, s^t and l^t, and L_a^t and L_o^t as functions of τ^t, p, π^t, h^{t-1}, q, w and R. The effects on e^t, n^t, s^t and l^t of changes in τ^t, p, π^t, h^{t-1}, q, w and R turn out to be ambiguous. However, we can get a rough idea of the various effects in action by considering the signs of the first partial derivatives of $D_i(.)$ with respect to its arguments, D_{ij} - where j denotes an argument of $D_i(.)$.[6]

We start with the analysis of D_e. It can be shown after some tedious calculations that $D_{e\pi} > 0$. Therefore, a marginal increase in longevity leads ceteris paribus to higher levels of education. The reason is that higher longevity implies a higher need for consumption and therefore income, which can be increased by raising the level of education and thereby productivity. In addition, $D_{ew} < 0$

[6]For completeness, these partial derivatives can be found in the Appendix, where we have normalised $w = 1$ to simplify the exposition. Note that the interplay of all these partial derivatives for any i and the relevant partial derivatives of L_a^t and L_o^t is what determines the (net) effects on e^t, n^t, s^t and l^t of marginal changes in τ^t, p, π^t, h^{t-1} and q. The exact nature of this interplay is of course determined by the Implicit Function Theorem.

represents the expected negative income effect of higher wages on education. Moreover, the sign of D_{ej} where $j = h^{t-1}$, L_a^t, L_o^t is ambiguous.[7] This follows from the existence of two opposing effects on e^t of a marginal increase in j. On the one hand, an increase in j implies that the benefit in terms of higher consumption from a marginal increase in the level of education is now higher and thus the agent has an incentive to spend more time on education. An increase in j, however, implies also that for any given level of education the agent enjoys now higher consumption and thereby she has can maintain her previous level(s) of consumption at a lower level of education.

Concerning the decision on n^t, we have that $D_{nq} < 0$. Therefore, a marginal increase in the cost of raising children as a proportion of wages leads ceteris paribus to less children for the obvious reasons. In addition, $D_{nL_a} > 0$ and $D_{nw} > 0$. An increase in L_a^t or w implies higher consumption levels and thereby the agent can afford more children.[8] However, the sign of D_{nj} where $j = e^t$, h^{t-1} is ambiguous. This ambiguity is a direct consequence of the existence of two opposing effects on n^t of a marginal increase in the level of period t human capital h^t, which in turn depends positively on both e^t and h^{t-1}. An increase in human capital implies higher labour income and that the agent can afford to have more children. An increase in h^t, on the other hand, implies also higher wages and therefore that the cost of upbringing is higher; the typical agent t has an incentive to decrease the size of her offspring.

We turn to the examination of D_s. It follows directly from (9) that $D_{s\pi} > 0$, $D_{sL_a} > 0$ and $D_{sL_o} < 0$. Higher longevity dictates a need for higher savings to finance future consumption, and thereby has a positive effect ceteris paribus on savings when adult. $D_{sL_o} < 0$ highlights the standard (negative) income effect on savings of a higher level of consumption when old. Moreover, $D_{sL_a} > 0$ represents the standard (positive) income effect on savings of a higher level of consumption when young. In addition, $D_{sR} > 0$ which reflects the standard substitution effect of higher interest rates. Nevertheless, the sign of D_{sj} where $j = e^t$, h^{t-1}, w is ambiguous. This ambiguity follows directly from the fact that higher j induces higher wages and thereby higher consumption both when adult and when old.

Finally, in analysing the decision of the typical agent t concerning the fraction of time spent on working as old we have that $D_{lp} < 0$, $D_{lL_o} < 0$ and $D_{lw} < 0$. That is, as intuition would suggest, higher pensions imply ceteris paribus a higher cost from working and therefore early retirement. Furthermore, $D_{lL_o} < 0$ and $D_{lw} < 0$ reflect the standard (negative) income effect on labour supply of higher wealth.[9] Notwithstanding, the sign of D_{lj} where $j = e^t$, h^{t-1} is ambiguous. This ambiguity follows directly from the observation that higher j induces higher wages and the standard ambiguous effect of higher wages on labour supply.

[7] When L_a^t and L_o^t can change is obvious from the two equations above.

[8] Note that the partial effect of q on n^t is reinforced through the effect of q on L_a^t.

[9] Note that the partial effect of p on l^t is reinforced through the effect of p on L_o^t.

We now turn to the investigation of our equilibria. Using the fact that in equilibrium $\overline{h}^t = h^t$ we have that the equilibrium paths for e^t, c_a^t, c_o^t, n^t, s^t, l^t, h^t, π^t, τ^t follow from (7)-(10), (2)-(5), and

$$\pi^t = \pi\left(h^t\right) \tag{17}$$

as a function of $n^{t-1}, s^{t-1}, l^{t-1}, h^{t-1}, \pi^{t-1}$ and $I \equiv \{p, a, q, w, R\}$. Accordingly, the equilibrium paths for n^t, s^t, l^t, h^t, π^t are given by the solution to a system of 5 autonomous difference equations. The characteristics of this dynamic system will be the focus of our analysis hereafter.

3.1 Stability Analysis

In order to analyse the dynamic system in question we first need the following: Using (2)-(4), (15) and (16) to substitute for H^t, c_a^t, c_o^t, L_a^t and L_o^t and recalling that H_2^t is a function of h^{t-1} and e^t we have that the household t's FOCs (7)-(10) can be re-written as

$$F^e(y^t; x^{t-1}, I) \equiv \hat{D}^e(n^t, s^t, l^t, h^t, \pi^t, e^t, \tau^t; q, p, w, R) = 0 \tag{18}$$

$$F^n(y^t; x^{t-1}, I) \equiv \hat{D}^n(n^t, s^t, h^t, \tau^t; q, w, R) = 0 \tag{19}$$

$$F^s(y^t; x^{t-1}, I) \equiv \hat{D}^s(n^t, s^t, l^t, h^t, \pi^t, \tau^t; q, p, w, R) = 0 \tag{20}$$

$$F^l(y^t; x^{t-1}, I) \equiv \hat{D}^l(s^t, l^t, h^t; p, w, R) = 0, \tag{21}$$

where $\hat{D}^i$ denotes the welfare gain from a marginal increase in i on the part of the typical agent t as a function of y^t and I, and $y^t = [x^t, e^t, \tau^t]$, $x^t = [n^t, s^t, l^t, h^t, \pi^t]$. Defining $T^t(.) = \frac{h^{t-1}\left[s^{t-1}R\left(1-\pi^{t-1}\right)-p\left(1-l^{t-1}\right)\pi^{t-1}\right]}{n^{t-1}}$, (5) can also be re-written as

$$F^\tau(y^t; x^{t-1}, I) \equiv \tau^t - [T^t(x^{t-1}; p, R)/h^t] + an^t = 0. \tag{22}$$

Finally we re-write (2) and (17) as

$$F^h(y^t; x^{t-1}, I) \equiv h^t - H(h^{t-1}, e^t) = 0 \tag{23}$$

and

$$F^\pi(y^t; x^{t-1}, I) \equiv \pi^t - \pi(h^t) = 0. \tag{24}$$

Assuming for the time being that the system (18)-(24) can be solved (at least locally), we have that the system in question gives implicitly the path $y^t = f(x^{t-1}; I)$, where f is a vector-function that belongs to $\mathcal{R}^7$ with components f^y each of which is a map $\mathcal{R}^5 \rightarrow \mathcal{R}$ - given I. Accordingly, the dynamic system we are interested in is part of the above equilibrium path: $x^t = f(x^{t-1}; I)$.

In what follows, to simplify the exposition we normalise by setting $w = 1$ and we deploy the following notations:

$$S_a \equiv -\frac{\phi''(c_a^t)c_a^t}{\phi'(c_a^t)}$$

$$S_o \equiv -\frac{\phi''(c_o^t)c_o^t}{\Phi'(c_o^t)}$$

$$S_\pi \equiv \frac{\pi'(h^t)h^t}{\pi(h^t)}$$

$$\Phi_a \equiv \delta\phi'(c_a^t)$$

$$\Phi_o \equiv \varphi\phi'(c_o^t)$$

$$N \equiv \beta\phi''(n^t)$$

$$E \equiv U''(1-e^t)$$

$$\Lambda \equiv \gamma U''(1-l^t)$$

$$T_n \equiv -\frac{T^t(.)}{n^{t-1}}$$

$$T_s \equiv \frac{T^t(.)}{s^{t-1}} + \frac{(1-l^{t-1})\pi^{t-1}h^{t-1}p}{n^{t-1}s^{t-1}}$$

$$T_l \equiv -\frac{T^t(.)}{(1-l^{t-1})} + \frac{(1-\pi^{t-1})s^{t-1}h^{t-1}R}{n^{t-1}(1-l^{t-1})}$$

$$T_h \equiv \frac{T^t(.)}{h^{t-1}}$$

$$T_\pi \equiv -\frac{T^t(.)}{\pi^{t-1}} + \frac{s^{t-1}R}{\pi^{t-1}}$$

$$L \equiv L_o^t + RL_a^t.$$

Moreover, we suppress, whenever there is no risk of confusion and until further notice, the dependence of all the relevant variables on time t. Finally, we note from (9) that

$$\Phi_2\pi R = \Phi_2.$$

It follows, then, that the Jacobian of the system (18)-(24) evaluated at the equilibrium, P, is given by the following 7×7 matrix

$$P = \begin{bmatrix} P_{11} & P_{12} \\ P_{21} & P_{22} \end{bmatrix},$$

$$P_{11} = \begin{bmatrix} F_n^n & F_s^n & F_l^n & F_h^n & F_\pi^n \\ F_n^s & . & & & . \\ F_n^l & & . & & . \\ F_n^h & & & . & . \\ F_n^\pi & . & . & . & F_\pi^\pi \end{bmatrix}$$

$$P_{12} = \begin{bmatrix} F_e^n & F_\tau^n \\ F_e^s & F_\tau^s \\ F_e^l & F_\tau^l \\ F_e^h & F_\tau^h \\ F_e^\pi & F_\tau^\pi \end{bmatrix}, \; P_{22} = \begin{bmatrix} F_e^e & F_\tau^e \\ F_e^\tau & F_\tau^\tau \end{bmatrix}$$

$$P_{21} = \begin{bmatrix} F_n^e & F_s^e & F_l^e & F_h^e & F_\pi^e \\ F_n^\tau & F_s^\tau & F_l^\tau & F_h^\tau & F_\pi^\tau \end{bmatrix},$$

where F_j^i denotes the first partial derivative of the function $F^i(.;.)$ with respect to j and $i, j = n, s, l, h, \pi, e, \tau$, and

$$F_l^n = F_\pi^n = F_e^n = 0,$$

$$F_e^s = 0,$$

$$F_n^l = F_\pi^l = F_e^l = F_\tau^l = 0,$$

$$F_n^h = F_s^h = F_l^h = F_\pi^h = F_\tau^h = 0,$$

$$F_n^\pi = F_s^\pi = F_l^\pi = F_e^\pi = F_\tau^\pi = 0,$$

$$F_s^\tau = F_l^\tau = F_\pi^\tau = F_e^\tau = 0,$$

$$F_n^n = N + HF_n^s, \; F_s^n = HF_n^s, \; F_h^n = -q\Phi_a(1-S_a), \; F_\tau^n = -HF_n^s,$$

$$F_n^s = -\frac{qS_a\Phi_a}{L_a}, \; F_s^s = -\frac{(S_aL_o + S_oRL_a)\Phi_a}{L_aL_o}, \; F_l^s = -\frac{(1-p)S_o\Phi_a}{L_o},$$

$$F_h^s = \frac{(S_a - S_o)\Phi_a}{H}, \; F_\pi^s = \frac{\Phi_a}{\pi}, \; F_\tau^s = -\frac{F_n^s}{q},$$

$$F_s^l = F_l^s(H/\pi), \; F_l^l = \Lambda + F_s^l((1-p)/R), \; F_h^l = \frac{(1-p)(1-S_o)\Phi_a}{\pi R},$$

$$F_h^h = 1, \; F_e^h = -H_2,$$

$$F_h^\pi = -\frac{\pi S_\pi}{H}, \; F_\pi^\pi = 1,$$

$$F_n^e = F_h^n H_2, \; F_s^e = F_h^s HH_2, \; F_l^e = F_h^l \pi H_2, \; F_h^e = -\frac{H_2\Phi_1(S_aRL_a + S_oL_o)}{HR},$$

$$F_\pi^e = \frac{H_2L_a\Phi_a}{\pi R}, \; F_e^e = E + \frac{\Phi_a LH_{22}}{R}, \; F_\tau^e = -F_n^e/q,$$

$$F_n^\tau = a,\ F_h^\tau = \frac{T}{H^2},\ F_\tau^\tau = 1.$$

Note that the determinant of P is non-zero and therefore P^{-1} exists and the Implicit Function Theorem can be applied to determine implicitly $y^t = f(x^{t-1}; I)$ from the system (18)-(24) and thereby our autonomous dynamic system $x^t = f(x^{t-1}; I)$. In particular,

$$P^{-1} = \begin{bmatrix} \Delta^{-1} & -\Delta^{-1} P_{12} P_{22}^{-1} \\ -P_{22}^{-1} P_{21} \Delta^{-1} & P_{22}^{-1} + P_{22}^{-1} P_{21} \Delta^{-1} P_{12} P_{22}^{-1} \end{bmatrix},$$

with

$$\begin{aligned} \Delta &= P_{11} - P_{12} P_{22}^{-1} P_{21} \\ &= \begin{bmatrix} \Delta_1^\tau \\ \Delta_2^\tau \\ \Delta_3^\tau \\ \Delta_4^\tau \\ \Delta_5^\tau \end{bmatrix} \end{aligned}$$

and[10]

$$\begin{aligned}
\Delta_1 &= \begin{bmatrix} N - HqS_a \frac{\Phi_a}{L_a} - HqS_a \frac{\Phi_a}{L_a} a \\ -HqS_a \frac{\Phi_a}{L_a} \\ 0 \\ -q\Phi_a \left(1 - S_a\right) - \frac{1}{H} qS_a \frac{\Phi_a}{L_a} T \\ 0 \end{bmatrix}, \\
\Delta_2 &= \begin{bmatrix} -qS_a \frac{\Phi_a}{L_a} - S_a \frac{\Phi_a}{L_a} a \\ -\left(S_a L_o + S_o R L_a\right) \frac{\Phi_a}{L_a L_o} \\ -\left(1-p\right) S_o \frac{\Phi_a}{L_o} \\ \left(S_a - S_o\right) \frac{\Phi_a}{H} - S_a \frac{\Phi_a}{L_a} \frac{T}{H^2} \\ \frac{\Phi_a}{\pi} \end{bmatrix}, \\
\Delta_3 &= \begin{bmatrix} 0 \\ -\left(1-p\right) S_o \Phi_a \frac{H}{L_o \pi} \\ \Lambda - \left(1-p\right)^2 S_o \Phi_a \frac{H}{L_o \pi R} \\ \left(1-p\right)\left(1-S_o\right) \frac{\Phi_a}{\pi R} \\ 0 \end{bmatrix}, \\
\Delta_4 &= \begin{bmatrix} -\frac{H_2^2}{ER+\Phi_a L H_{22}} Rq\Phi_a \left(1-S_a\right) - \Phi_a \left(1-S_a\right) H_2^2 \frac{R}{ER+\Phi_a L H_{22}} a \\ \frac{H_2^2}{ER+\Phi_a L H_{22}} R \left(S_a - S_o\right) \Phi_a \\ \frac{H_2^2}{ER+\Phi_a L H_{22}} \left(1-p\right)\left(1-S_o\right) \Phi_a \\ 1 - \frac{H_2^2}{ER+\Phi_a L H_{22}} \Phi_1 \frac{S_a R L_a + S_o L_o}{H} - \Phi_a \left(1-S_a\right) H_2^2 \frac{R}{ER+\Phi_a L H_{22}} \frac{T}{H^2} \\ \frac{H_2^2}{ER+\Phi_a L H_{22}} L_a \frac{\Phi_a}{\pi} \end{bmatrix},
\end{aligned}$$

[10] A^τ denotes the transpose of a matrix A.

$$\Delta_5 = \begin{bmatrix} 0 \\ 0 \\ 0 \\ -\pi \frac{S_\pi}{H} \\ 1 \end{bmatrix}.$$

Moreover,[11] after defining f_x^y as the 7×5 matrix with i row the partial gradient of $f^i(x^{t-1}; I)$ with respect to x^{t-1} and G as the 7×5 matrix with i column the partial gradient of $F^i(y^t; x^{t-1}, I)$ with respect to x^{t-1}, where $i = n$, s, l, h, π, e, τ we have that

$$f_x^y = P^{-1}G,$$

$$\begin{aligned} G &= \begin{bmatrix} G_1 \\ G_2 \end{bmatrix}, \\ G_1 &= \begin{bmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & H_1 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}, \\ G_2 &= \begin{bmatrix} 0 & 0 & 0 & -\frac{H_{21}\Phi_a L}{R} & 0 \\ T_n/H & T_s/H & T_l/H & T_h/H & T_\pi/H \end{bmatrix}. \end{aligned}$$

Hence, the 5×5 Jacobian matrix evaluated at the equilibrium, J, of the dynamic system $x^t = f(x^{t-1}; I)$ is given by

$$\begin{aligned} J &= \begin{bmatrix} \Delta^{-1} & -\Delta^{-1}P_{12}P_{22}^{-1} \end{bmatrix} G \\ &= \Delta^{-1}G_1 - \Delta^{-1}P_{12}P_{22}^{-1}G_2 \\ &= \Delta^{-1}(G_1 - P_{12}P_{22}^{-1}G_2) \\ &\equiv \Delta^{-1}\bar{G}. \end{aligned}$$

Our aim is to determine the eigenvalues of J and therefore of the matrix $\Delta^{-1}\bar{G}$. To do so we first need to calculate $\bar{G}$. We have that

$$\bar{G} = \begin{bmatrix} K_n & K_s & K_l & K_h & K_\pi \\ \frac{K_n}{qH} & \frac{K_s}{qH} & \frac{K_l}{qH} & \frac{K_h}{qH} & \frac{K_\pi}{qH} \\ 0 & 0 & 0 & 0 & 0 \\ M_n & M_s & M_l & M_h & M_\pi \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

[11] The existence of P^{-1} follows from the existence of P_{22}^{-1} and of Δ^{-1}. The existence of Δ^{-1} has been checked by using Scientific WorkPlace 3.1.

where $K_i = -qS_a \frac{\Phi_a}{L_a} T_i$, and $M_j = -\Phi_a (1 - S_a) H_2^2 \frac{R}{ER+\Phi_a LH_{22}} \frac{T_j}{H}$, $i = n, s, l, h, \pi$, $j = n, s, l, \pi$, and $M_h = H_1 - \frac{H_2}{ER+\Phi_a LH_{22}} H_{21} \Phi_a L + \Phi_a (-1 + S_a) H_2^2 \frac{R}{ER+\Phi_a LH_{22}} \frac{T_h}{H}$. Clearly, then, $rank(\bar{G}) = 2$ and $|\bar{G}| = 0$.[12]

Therefore, $|J| = 0$ and thus the Jacobian matrix of the dynamic system we are interested in evaluated at the steady state - assuming for the time being that a steady state exists - is singular. Hence, there is no point in linearising the dynamic system in question around the steady state and analysing the qualitative characteristics of the resulting linear system, simply because this linear system and the original non-linear dynamic system are not topologically equivalent.[13] Yet, we are still able, as it turns out, to characterise the stability of the non-linear dynamic system $x^t = f(x^{t-1}; I)$:

Theorem 1 *Let x^o be a steady state of $x^t = f(x^{t-1}; I)$. x^o is hyperbolic and asymptotically stable for any I.*

Proof. The proof rests on showing that the eigenvalues of the matrix $\Delta^{-1}\bar{G}$ are all zero for any I. By definition λ is an eigenvalue of the matrix in question if there exists 5−vector $c \neq 0_{5\times 1}$ such that $(\Delta^{-1}\bar{G})c = \lambda c$, or, equivalently, such that $\bar{G}c = \lambda \Delta c$. Clearly, then, given that $|\bar{G}| = 0$, $\lambda = 0$ is an eigenvalue.

Note, given $|\Delta| \neq 0$, that if $c \neq 0_{5\times 1}$ then $\Delta c \neq 0_{5\times 1}$. Therefore, if there is a non-zero eigenvalue λ then it must be that the corresponding eigenvector c is a solution to the system $\bar{G}c = b$ where $b \equiv \lambda \Delta c \neq 0_{5\times 1}$ is a 5−vector. Hence, finding whether there exist non-zero eigenvalues of $\Delta^{-1}\bar{G}$ amounts to finding whether there exists a solution $c \neq 0_{5\times 1}$ to the system $\bar{G}c = b$.

Define as Z the matrix $\begin{bmatrix} 1 & -qH & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix}$ and note that Z^{-1} exists.[14]

Accordingly the system $\bar{G}c = b$ has a solution $c \neq 0_{5\times 1}$ if and only if c is a solution of $Z\bar{G}c = Zb$. Note that $Z\bar{G} = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 \\ \frac{K_n}{qH} & \frac{K_s}{qH} & \frac{K_l}{qH} & \frac{K_h}{qH} & \frac{K_\pi}{qH} \\ 0 & 0 & 0 & 0 & 0 \\ M_n & M_s & M_l & M_h & M_\pi \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$ and thus

[12] Note that the first column is equal to the third column times $\frac{T_n}{T_l}$, the second column is equal to the third column times $\frac{T_s}{T_l}$ and the fifth column is equal to the third column times $\frac{T_\pi}{T_l}$.

[13] Later on in the paper we resort to numerical simulations in order to get an idea of how the system behaves over time.

[14] Z is constructed from the kernel basis of $\bar{G}^\tau$. The null space (or kernel) of a matrix A is the vector space consisting of all solutions of the homogenous equations $Az = \mathbf{0}$. It is obvious that the nullspace basis of $\bar{G}^\tau$ consists of the first, third and fifth rows of Z.

$c \neq 0_{5\times1}$ is a solution of $Z\bar{G}c = Zb$ if and only if $Z^B b = 0_{3\times1}$ and $\bar{G}^B c = \lambda\Delta^B c$, where $Z^B = \begin{bmatrix} 1 & -qH & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix}$, $\bar{G}^B = \begin{bmatrix} \frac{K_n}{qH} & \frac{K_s}{qH} & \frac{K_l}{qH} & \frac{K_h}{qH} & \frac{K_\pi}{qH} \\ M_n & M_s & M_l & M_h & M_\pi \end{bmatrix}$ and $\Delta^B = \begin{bmatrix} \Delta_2^\tau \\ \Delta_4^\tau \end{bmatrix}$. Equivalently, and making use of the definition for b and the fact that $\lambda \neq 0$, we have that there exists a solution $c \neq 0_{5\times1}$ to the system $\bar{G}c = b$ if and only if $c \neq 0_{5\times1}$ is the solution to the homogenous system $\Psi c = 0_{5\times1}$ where $\Psi = \begin{bmatrix} Z^B\Delta \\ \bar{G}^B - \lambda\Delta^B \end{bmatrix}$. However, $rank(\Psi) = 5$ and thus the only solution to the system $\Psi z = 0_{5\times1}$ is the trivial one.[15] ■

Thus, if a steady state exists then it is hyperbolic and a sink for any I. The fact that the steady state is hyperbolic implies that bifurcations of equilibria when I changes cannot occur. That is, when structural parameters change there is no change in the number of steady states, their stability type and the nature of the orbits near a given steady state.

In fact, there is a uniquely determined steady state for any I. To see this note that the steady state is given by

$$f^x(x^o, I) - x^o = 0.$$

By the implicit function theorem we know that if the determinant of the Jacobian of the above system $\bar{J}$ is non-singular then the above system determines uniquely the steady state x^o as a function of the model's parameters in some neighbourhood of I. But, an eigenvalue of $\bar{J}$, μ, is equal to $1 - \lambda$, where λ is an eigenvalue of J. Therefore, $\mu = -1$ has multiplicity 5 and $\left|\bar{J}\right| \neq 0$ for any I.

The asymptotic stability of the unique steady state, on the other hand, implies that any transition path to the steady state is a short-run equilibrium! Accordingly, our model supports a plethora of short-run dynamic paths to a uniquely defined long-run equilibrium depending on the functional forms, parameters and initial conditions.

Our results come in striking contrast to the existing ones, in particular to the result of Becker et al. (1990) where longevity is not endogenous. In more detail, Becker et al. (1990) find multiple long-run equilibria (in particular one unstable and two locally stable), whilst we find a unique globally stable long-run equilibrium. Our results also differ from those of other models with endogenous longevity. Blackburn and Cipriani (2000) and Lagerlöf (2000) in fact, find multiple development regimes where the limiting outcomes of the economy are non ergodic but depend crucially on initial conditions.

The next step, in principle, would be to attempt to characterise in more detail the long-run equilibrium. However, without further constraints on preferences, technology and parameters we cannot - as it was expected - derive firm

[15]That $rank(\Psi) = 5$ has been checked by using Scientific WorkPlace 3.1.

conclusions from a comparative statics analysis of the steady state. For this reason, in what follows we provide some numerical simulations of our model.[16]

4 Simulations

To facilitate our simulations we assume logarithmic preferences and the following logistic form for the survival function:

$$\pi^t = \frac{\underline{\pi}\left[\exp\left(\xi\nu\right)+1\right]+\overline{\pi}\left[\exp\left(\xi h^t\right)-1\right]}{\exp\left(\xi\nu\right)+\exp\left(\xi h^t\right)} \tag{25}$$

where ν is a parameter determining the turning point, ξ determines the speed of transition and $\underline{\pi}$ and $\overline{\pi}$ represent the lower and upper bound of the survival probability. The human capital production function is specified as:

$$h^t = h^{t-1}\left(1+\mu e^t\right) \tag{26}$$

In what follows denote with $\bar{q} = q + a$. The parameter $\overline{q}$ is the exogenous component of the child rearing costs. The parameter values are chosen quite arbitrarily, but do ensure interior solutions and give steady state results which are similar to those found in empirical studies.

Parameter	ξ	ν	$\underline{\pi}$	$\overline{\pi}$	$\hat{\beta}$	$\hat{\varphi}$	$\hat{\gamma}$
Value	0.02	0.3	0	1	1.69	3.12	0.39

Parameter	δ	$\overline{q}$	a	p	R	μ	w
Value	0.8	0.3	0	0.65	4.5	1.6	1

The initial population consists of one adult and the other initial conditions are $h^0 = 0.5$ and $\tau^1 = 0$. Each of the three periods correspond to 25 years. While e^t and π^t are obviously always increasing during the transition to the steady state, the resulting paths for n^t, s^t, l^t and τ^t are non-monotonic.

Fig. 1 plots the diagram of n^t during the transition. Here, and in all the other diagrams, we assume that our economy starts in 1450. After dropping the initial 6 periods to get the model up and running, we plot the remaining observations from 1600 until 2075. Fertility starts from a low level, then increases, and then falls again until it reaches the steady state like in the actual demographic transition. The length of the phase of increasing fertility is affected by the parameter $\hat{\beta}$: a lower value of $\hat{\beta}$ reduces the duration of the baby boom. The extent of the boom is affected by $\overline{q}$: when it falls the boom flattens and may become insignificant. Hence a country with low cost of child rearing may not experience the same demographic transition and move from a state of high fertility to low fertility, which is particularly true for many less developed

[16]However, in the Appendix, we do report - for completeness - the matrix with the comparative statics on the steady state (and the steps which are needed to arrive at this matrix).

countries. The fertility path in our simulation increases from about 1.1 at the beginning of the eighteenth century (which corresponds to an annual population growth rate of 0.47%) to reach 1.8 at the end of the nineteenth century (2.5% per annum) to fall to about 1 today (i.e. the population replacement level).[17] The path of the life expectancy is shown in Fig. 2. It is very similar to the actual experience of Europe and the USA (see Kremer, 1993): from an initial long phase of very small improvements, it begins to increase fast in the last half of the nineteenth century to reach its steady state value of 75.

In Fig. 3 we plot the time path for s^t which first increases and then falls toward its steady state value. We allow for s to be negative, that is borrowing at the world interest rate can take place and negative bequests are not ruled out. This simplification is counterbalanced by the fact that positive unintended bequests are also absorbed by the Government. The saving rate becomes positive when growth takes off and it increases until the last quarter of the twentieth century before falling to its the steady state value of 0.28. The opposite path is found for l^t in Fig. 4. Note that a decrease in l^t corresponds to a decrease in the retirement age, and vice versa. After an initial long fall in the retirement age, like that experienced by many developed countries, there is a slight increase to the steady state. Finally the tax rate, $-\tau^t$, increases initially because of the negative bequests, then because population starts ageing and the PAYG system begins to operate, then falls because of the effect of the positive unintended bequests[18] and then increases again to the steady state as in Fig. 5.

The transitional path of the per capita growth rate is displayed in Fig. 6. It starts from 0.11 (that is 0.43% per year) in 1800, it increases to 0.41 (1.38% per year) in 1900 and reaches a maximum of 1.14 (3.08% per year) around 1975 to fall to 0.93 (2.66% per year) at the steady state. This path matches very closely that of many currently developed countries (see for example Barro and Sala-i-Martin, 1995).

Fig. 7 plots the dependency ratio over time. Because the initial decrease in fertility rates more than offsets the initial increase in the number of old agents dependency initially falls. It then increases, together with the temporary increase in fertility and falls again after the baby boom. Finally it raises to its stationary level. In the last diagrams we plot two ageing indices: the ratio of old agents to total population and to young agents in Fig. 8 and Fig. 9 respectively. Both are increasing with time until they reach the steady state and closely match the actual paths and forecasts in developed countries (see for example Eurostat, 1995).

[17] The fertility rate in this paper is the number of children per person. Therefore it coincides with 1+population growth rate over 25 years. The actual European population growth rate is similar to our simulated path, which however overshoots actual fertility at its peak by about 1% (see Kremer, 1993).

[18] To see this note that if the model is simulated with a 0% tax rate on bequests (in which case the implicit assumption is that unintented bequests are 'wasted') the time path of τ is always increasing until it becomes stable. It therefore follows the same path as the ratio of old agents to adults.

The following tables present some comparative statics results, where $G = n(1 + \mu e) - 1$ is the steady state growth rate of total output and $g = \mu e$ is the per capita growth rate in the steady state. *old/pop* is the ratio of old agents to total population and *dep* is the dependency ratio which includes the children too. The last two columns of the second table show the old to young agents ratio and the old agents to adults ratio.

	e	n	s	l	$-\tau$	G	g
Baseline	0.572	1.002	0.279	0.284	0.242	2.64%	2.63%
$a = 0.1$		1.411	0.248	0.328	0.303	4.06%	
$\mu = 1.7$	0.582	1.011	0.283	0.278	0.233	2.83%	2.79%
$\delta = 0.85$	0.604	0.963	0.289	0.269	0.251	2.59%	2.74%
$\hat{\varphi} = 3.91$	0.622	0.909	0.333	0.351	0.233	2.41%	2.80%
$\hat{\beta} = 1.87$		1.103	0.274	0.291	0.218	3.04%	
$\overline{q} = 0.25$		1.252	0.299	0.254	0.202	3.56%	
$\hat{\gamma} = 0.47$		0.942	0.259	0.193	0.291	2.39%	
$p = 0.5$		1.116	0.336	0.442	0.130	3.09%	

	dep	*old/pop*	*old/yng*	*old/adl*
Baseline	2.000	0.333	0.996	0.998
$a = 0.1$	2.120	0.227	0.502	0.709
$\mu = 1.7$	2.000	0.330	0.978	0.989
$\delta = 0.85$	2.001	0.346	1.079	1.039
$\hat{\varphi} = 3.91$	2.009	0.366	1.211	1.100
$\hat{\beta} = 1.87$	2.010	0.301	0.821	0.906
$\overline{q} = 0.25$	2.051	0.262	0.638	0.799
$\hat{\gamma} = 0.47$	2.004	0.353	1.127	1.062
$p = 0.5$	2.012	0.297	0.802	0.896

The first row shows the steady state results for our baseline values of the parameters. Agents spend 14.3 years in education, the population growth rate is near the replacement level and the savings rate is 28%. The retirement age is 57 and the tax rate is 24%.[19] In the second line the effect of a child rearing subsidy is considered. This subsidy increases the fertility rate, thus decreasing the old age dependency ratio and it increases the growth rate of total output. It has also the effect of decreasing the saving rate and increasing the retirement age. Similarly, a less generous pension scheme ($p = 0.5$) or a fall in the exogenous component of child rearing costs ($\overline{q} = 0.25$), increases growth and the fertility rate, and also decreases the tax rate and the old agents to adults ratio. However, while both events increase the saving rate, the former increases the retirement age, while the latter reduces it. If the human capital technology becomes more efficient ($\mu = 1.7$) per capita growth obviously increases and so does the fertility

[19]These numbers are derived considering that each period lasts 25 years. Then 0.572 corresponds to 14.3 years and the retirement age is 50+0.28*25.

rate and the saving rate. Here an income effect acts on the retirement age reducing it. As far as preferences are concerned, the table shows the effects of a change in the discount rate (δ), in the preference for old age consumption ($\hat{\varphi}$), children ($\hat{\beta}$) and leisure in old age ($\hat{\gamma}$). A higher discount rate decreases fertility and the retirement age while increasing the savings rate and the per capita growth rate. The same results follow an increase in the preference for old age consumption, except that the effect on the retirement age goes in the opposite direction. A higher taste for leisure when old has the obvious effect of inducing early retirement, but it also reduces fertility and the saving rate.

Choosing a lower starting value of h^0 does increase the number of periods it takes to reach the steady state (which, of course, is not altered) but the time paths of the endogenous variables are eventually very similar to those displayed in Fig.1-8.

5 Conclusion

This paper is an attempt to study ageing in a model where demographics (fertility and longevity) and economic outcomes are determined jointly within a general equilibrium model. The few other existing models with fully endogenous demography have not been geared to study ageing alongside growth and their implications. Our analysis has been based on a simple OLG model in which life expectancy is endogenous and reproductive agents invest in education, produce and consume output. Retirement age is endogenous and when retired agents receive a PAYG pension. The equilibrium is characterised by a sink and therefore supports a plethora of short-run dynamic paths. Under specific assumptions for preferences and technology we have shown that the model yields predictions about the relationships between economic and demographic outcomes that can fit the stylised facts. In particular, a demographic transition from a situation of increasing income and increasing fertility to increasing income and falling fertility is derived. This transition and the subsequent ageing of the population is driven by the endogeneity of life expectancy. The last Section of the paper provides also the results of a couple of policy experiments concerning the pension scheme and the child subsidy scheme.

There are a number of ways in which the model could be extended and improved. Ideally one should better consider the role of bequests which in this paper are only taken into account as unintended bequests (taxed at 100% by the government). Together with this, a more complete model of ageing should incorporate another important reason to have children: the old age support motive. Analysing intergenerational transfers in both ways would however significantly complicate the present model and would probably be of not much interest in modern developed countries. Finally, we would also like to incorporate physical, as well as human, capital and relax in that way the small open economy assumption.

6 References

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7 Appendix

7.1 First Partial Derivatives of $D_i(.)$

$$
\begin{aligned}
D_{ee} &= H^t_{22}[\delta L^t_1\phi'(c^t_a)+L^t_2\varphi\pi^t\phi'(c^t_o)]+H^{t2}_2[\delta L^{t2}_1\phi''(c^t_a)+L^{t2}_2\varphi\pi^t\phi''(c^t_o)]\\
&\quad +U''(1-e^t),\\
D_{e\pi} &= H^t_2 L^t_2\varphi\phi'(c^t_o),\\
D_{eL_a} &= H^t_2\delta[L^t_1 H^t\phi''(c^t_a)+\phi'(c^t_a)],\\
D_{eL_o} &= H^t_2\varphi\pi^t[L^t_2 H^t\phi''(c^t_o)+\phi'(c^t_o)],\\
D_{eh^{t-1}} &= H^t_{21}[\delta L^t_1\phi'(c^t_a)+L^t_2\varphi\pi^t\phi'(c^t_o)]+H^t_2 H^t_1[\delta L^{t2}_1\phi''(c^t_a)+L^{t2}_2\varphi\pi^t\phi''(c^t_o)],\\
D_{ew} &= H^t_2 H^t[\delta L^{t2}_1\phi''(c^t_a)+L^{t2}_2\varphi\pi^t\phi''(c^t_o)].
\end{aligned}
$$

$$
\begin{aligned}
D_{ne} &= -H^t_2\delta q[\phi'(c^t_a)+H^t L^t_1\phi''(c^t_a)],\\
D_{nq} &= -H^t\delta\phi'(c^t_a),\\
D_{nL_a} &= -H^{t2}\delta q\phi''(c^t_a),\\
D_{nn} &= \beta\phi''(n^t),\\
D_{nh^{t-1}} &= -\delta q H^t_1[\phi'(c^t_a)+H^t L^t_1\phi''(c^t_a)],\\
D_{nw} &= -H^{t2}q[\delta L^t_1\phi''(c^t_a)].
\end{aligned}
$$

$$
\begin{aligned}
D_{se} &= H^t_2[\varphi\pi^t R L^t_2\phi''(c^t_o)-\delta L^t_1\phi''(c^t_a)],\\
D_{s\pi} &= \varphi R\phi'(c^t_o),\\
D_{sL_a} &= -\delta H^t\phi''(c^t_a),\\
D_{sL_o} &= H^t\varphi\pi^t R\phi''(c^t_o),\\
D_{sh^{t-1}} &= H^t_1[\varphi\pi^t R L^t_2\phi''(c^t_o)-\delta L^t_1\phi''(c^t_a)],\\
D_{sw} &= H^t[\varphi\pi^t R L^t_2\phi''(c^t_o)-\delta L^t_1\phi''(c^t_a)],\\
D_{sR} &= \varphi\pi^t\phi'(c^t_o).
\end{aligned}
$$

$$
\begin{aligned}
D_{le} &= H^t_2\varphi(1-p)[\phi'(c^t_o)+H^t L^t_o\phi''(c^t_o)],\\
D_{ll} &= \gamma U''(1-l^t),\\
D_{lp} &= -\varphi H^t\phi'(c^t_o),\\
D_{lL_o} &= H^{t2}\varphi(1-p)\phi''(c^t_o),\\
D_{lh^{t-1}} &= H^t_1\varphi(1-p)[\phi'(c^t_o)+H^t L^t_o\phi''(c^t_o)],\\
D_{lw} &= \varphi(1-p)H^{t2}L^t_o\phi''(c^t_o)].
\end{aligned}
$$

7.2 Steady State Analysis

The long-run equilibrium $y^o=[x^o,e^o,\tau^o]$ is given implicitly by

$$F^y(y^o;x^o,I)\equiv\hat{F}^y(x^o,e^o,\tau^o;I)=0. \tag{27}$$

The Jacobian of the system (27), $\hat{P}$, is given by the following 7×7 matrix[20]

$$
\begin{aligned}
\hat{P} &= \begin{bmatrix} \hat{P}_{11} & \hat{P}_{12} \\ \hat{P}_{21} & \hat{P}_{22} \end{bmatrix}, \\
\hat{P}_{11} &= \begin{bmatrix} \hat{F}^n_n & \hat{F}^n_s & \hat{F}^n_l & \hat{F}^n_h & \hat{F}^n_\pi \\ \hat{F}^s_n & . & & & . \\ \hat{F}^l_n & & . & & . \\ \hat{F}^h_n & & & . & . \\ \hat{F}^\pi_n & . & . & . & \hat{F}^\pi_\pi \end{bmatrix} \\
\hat{P}_{12} &= \begin{bmatrix} \hat{F}^n_e & \hat{F}^n_\tau \\ \hat{F}^s_e & \hat{F}^s_\tau \\ \hat{F}^l_e & \hat{F}^l_\tau \\ \hat{F}^h_e & \hat{F}^h_\tau \\ \hat{F}^\pi_e & \hat{F}^\pi_\tau \end{bmatrix}, \ \hat{P}_{22} = \begin{bmatrix} \hat{F}^e_e & \hat{F}^e_\tau \\ \hat{F}^\tau_e & \hat{F}^\tau_\tau \end{bmatrix} \\
\hat{P}_{21} &= \begin{bmatrix} \hat{F}^e_n & \hat{F}^e_s & \hat{F}^e_l & \hat{F}^e_h & \hat{F}^e_\pi \\ \hat{F}^\tau_n & \hat{F}^\tau_s & \hat{F}^\tau_l & \hat{F}^\tau_h & \hat{F}^\tau_\pi \end{bmatrix},
\end{aligned}
$$

where $\hat{F}^i_j$ denotes the first partial derivative of the function $\hat{F}^i(.;.)$ with respect to j and $i, j = n, s, l, h, \pi, e, \tau$. Note that $\hat{F}^{i'} = F^{i'}$ for any $i' = n, s, l, \pi$ and thereby $\hat{F}^{i'}_j = F^{i'}_j$. Note also that $\hat{F}^h_e = F^h_e$, $\hat{F}^e_\kappa = F^e_\kappa$ for any $\kappa = n, s, l, \pi, e, \tau$ and, $\hat{F}^\tau_e = F^\tau_e$, $\hat{F}^\tau_\tau = F^\tau_\tau$. However,

$$\hat{F}^h_h = F^h_h - H_1,$$

$$\hat{F}^e_h = F^e_h + \frac{\Phi_a H_{21} L}{R},$$

$$\hat{F}^\tau_n = F^\tau_n + \hat{T}_n, \ \hat{F}^\tau_s = \hat{T}_s, \ \hat{F}^\tau_l = \hat{T}_l, \ \hat{F}^\tau_h = 0, \ \hat{F}^\tau_\pi = \hat{T}_\pi,$$

with

$$
\begin{aligned}
\hat{T}_n &\equiv -\frac{\hat{T}(.)}{n} \\
\hat{T} &\equiv \frac{sR - \pi(sR + p(1-l))}{n} \\
\hat{T}_s &\equiv -R\frac{(1-\pi)}{n} \\
\hat{T}_l &\equiv -\frac{\pi R}{n} \\
\hat{T}_\pi &\equiv \frac{sR + p(1-l)}{n}.
\end{aligned}
$$

[20]In what follows we suppress the superscript o whenever there is no risk for confusion.

Note that the determinant of $\hat{P}^{-1}$ exists. In more detail,

$$\hat{P}^{-1}=\begin{bmatrix} \hat{\Delta}^{-1} & -\hat{\Delta}^{-1}\hat{P}_{12}\hat{P}_{22}^{-1} \\ -\hat{P}_{22}^{-1}\hat{P}_{21}\hat{\Delta}^{-1} & \hat{P}_{22}^{-1}+\hat{P}_{22}^{-1}\hat{P}_{21}\hat{\Delta}^{-1}\hat{P}_{12}\hat{P}_{22}^{-1} \end{bmatrix},$$

with

$$\begin{aligned} \hat{\Delta} &= \hat{P}_{11}-\hat{P}_{12}\hat{P}_{22}^{-1}\hat{P}_{21} \\ &= \begin{bmatrix} \hat{\Delta}_1^{\tau} \\ \hat{\Delta}_2^{\tau} \\ \hat{\Delta}_3^{\tau} \\ \hat{\Delta}_4^{\tau} \\ \hat{\Delta}_5^{\tau} \end{bmatrix} \end{aligned}$$

and

$$\begin{aligned}
\hat{\Delta}_1 &= \begin{bmatrix} N-HqS_a\frac{\Phi_a}{L_a}-HqS_a\frac{\Phi_a}{L_a}(a+\hat{T}_n) \\ -HqS_a\frac{\Phi_a}{L_a}-HqS_a\frac{\Phi_a}{L_a}\hat{T}_s \\ -HqS_a\frac{\Phi_a}{L_a}\hat{T}_l \\ -q\Phi_a\left(1-S_a\right) \\ -HqS_a\frac{\Phi_a}{L_a}\hat{T}_\pi \end{bmatrix}, \\
\hat{\Delta}_2 &= \begin{bmatrix} -qS_a\frac{\Phi_a}{L_a}-S_a\frac{\Phi_a}{L_a}(a+\hat{T}_n) \\ -\left(S_aL_o+S_oRL_a\right)\frac{\Phi_a}{L_aL_o}-S_a\frac{\Phi_a}{L_a}\hat{T}_s \\ -\left(1-p\right)S_o\frac{\Phi_a}{L_o}-S_a\frac{\Phi_a}{L_a}\hat{T}_l \\ \left(S_a-S_o\right)\frac{\Phi_a}{H} \\ \frac{\Phi_a}{\pi}-S_a\frac{\Phi_a}{L_a}\hat{T}_\pi \end{bmatrix}, \\
\hat{\Delta}_3 &= \Delta_3, \\
\hat{\Delta}_4 &= \begin{bmatrix} -\frac{H_2^2}{ER+\Phi_aLH_{22}}Rq\Phi_a\left(1-S_a\right)-\Phi_a\left(1-S_a\right)H_2^2\frac{R}{ER+\Phi_aLH_{22}}(a+\hat{T}_n) \\ \frac{H_2^2}{ER+\Phi_aLH_{22}}R\left(S_a-S_o\right)\Phi_a-\Phi_a\left(1-S_a\right)H_2^2\frac{R}{ER+\Phi_aLH_{22}}\hat{T}_s \\ \frac{H_2^2}{ER+\Phi_aLH_{22}}\left(1-p\right)\left(1-S_o\right)\Phi_a-\Phi_a\left(1-S_a\right)H_2^2\frac{R}{ER+\Phi_aLH_{22}}\hat{T}_l \\ 1-H_1-H_2\frac{R}{ER+\Phi_aLH_{22}}\left(H_2\Phi_1\frac{S_aRL_a+S_oL_o}{HR}-H_{21}\Phi_1\frac{L}{R}\right) \\ \frac{H_2^2}{ER+\Phi_aLH_{22}}L_a\frac{\Phi_a}{\pi}-\Phi_a\left(1-S_a\right)H_2^2\frac{R}{ER+\Phi_aLH_{22}}\hat{T}_\pi \end{bmatrix}, \\
\hat{\Delta}_5 &= \Delta_5.
\end{aligned}$$

Moreover,[21] after defining f_I^y as the 7×5 matrix with i row the partial gradient of $f^i(x;I)$ with respect to I and Q as the 7×5 matrix with i column the partial

[21]The existence of $\hat{P}^{-1}$ follows from the existence of $\hat{P}_{22}^{-1}$ and of $\hat{\Delta}^{-1}$. The existence of $\hat{\Delta}^{-1}$ has been checked by using Scientific WorkPlace 3.1.

gradient of $\hat{F}^i(x,e,\tau,I)$ with respect to I, where $i = n,\ s,\ l,\ h,\ \pi,\ e,\ \tau$ we have that

$$f_I^y = \hat{P}^{-1}Q,$$

$$\begin{aligned}
Q &= \begin{bmatrix} Q_1 \\ Q_2 \end{bmatrix}, \\
Q_1 &= \begin{bmatrix} 0 & 0 & D_{nq} - nD_{nL_a} & D_{nw} & 0 \\ (1-l)D_{sL_o} & 0 & -nD_{sL_a} & D_{sw} & D_{sR} + sD_{sL_o} \\ D_{lp} + (1-l)D_{lL_o} & 0 & 0 & D_{lw} & sD_{lL_o} \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}, \\
Q_2 &= \begin{bmatrix} (1-l)D_{eL_o} & 0 & -nD_{eL_a} & D_{ew} & sD_{eL_o} \\ \frac{(1-l)\pi}{n} & n & 0 & 0 & -\frac{s(1-\pi)}{n} \end{bmatrix}.
\end{aligned}$$

Hence,

$$\begin{aligned}
f_I^y &= \begin{bmatrix} \hat{\Delta}^{-1} & -\hat{\Delta}^{-1}\hat{P}_{12}\hat{P}_{22}^{-1} \\ -\hat{P}_{22}^{-1}\hat{P}_{21}\hat{\Delta}^{-1} & \hat{P}_{22}^{-1} + \hat{P}_{22}^{-1}\hat{P}_{21}\hat{\Delta}^{-1}\hat{P}_{12}\hat{P}_{22}^{-1} \end{bmatrix} \begin{bmatrix} Q_1 \\ Q_2 \end{bmatrix} \\
&= \begin{bmatrix} \hat{\Delta}^{-1}Q_1 - \hat{\Delta}^{-1}\hat{P}_{12}\hat{P}_{22}^{-1}Q_2 \\ -\hat{P}_{22}^{-1}\hat{P}_{21}\hat{\Delta}^{-1} + (\hat{P}_{22}^{-1}\hat{P}_{21}\hat{\Delta}^{-1}\hat{P}_{12}\hat{P}_{22}^{-1})Q_2 \end{bmatrix} \\
&= \begin{bmatrix} \hat{\Delta}^{-1}(Q_1 - \hat{P}_{12}\hat{P}_{22}^{-1}Q_2) \\ -\hat{P}_{22}^{-1}\hat{P}_{21}\hat{\Delta}^{-1}(\mathbf{I} - \hat{P}_{12}\hat{P}_{22}^{-1}Q_2) \end{bmatrix} \\
&\equiv \begin{bmatrix} \hat{\Delta}^{-1}\bar{Q} \\ -\hat{P}_{22}^{-1}\hat{P}_{21}\hat{\Delta}^{-1}\tilde{Q} \end{bmatrix}.
\end{aligned}$$

The latter matrix provides us with the comparative statics on the steady state. However, without further constraints on preferences, technology and parameters we cannot derive firm conclusions on the signs of the elements of the matrix in question.

8 Figures

Figure 1: The path of fertility

Figure 2: The path of life expectancy

Figure 3: The path of the savings rate

Figure 4: The path of employment as old

Figure 5: The path of the tax rate

Figure 6: The path of the per capita growth rate

Figure 7: The dependency ratio

Figure 8: The ratio of old agents to total population

Figure 9: The ratio of old to young agents